STUDIA UNIVERSITATIS BABEȘ-BOLYAI
Engineering 68(1) 2023 DOI: 10.24193/subbeng.2023.1.6

Dynamic behavior of a clamped circular plate and strain energy representation (part I)

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Abstract. *The first part of the paper presents a study regarding the dynamics behavior of thin circular plate clamped all around. An analytical investigation is performed and the results in terms of mode shapes are used to highlight the plate's dynamics. The modal shapes are obtained using Bessel functions and their graphic representation is presented in 3D by using MS Excel software.*

Keywords: *circular plate, mode shape, Bessel functions*

1. Introduction

In mechanical and civil engineering structures, the circular plates are often used.

Their support types are imposed by different conditions and often are imposed by the structures functions and exploitation [1].

Circular plates are plane and thinned structures which are characterized by the thickness h. The thickness is small compared to the radius R [2].

Many researchers have been obtained analytical solutions. Their research has focus on the topic of natural frequencies of the plates [3-6].

The first researches regarding vibrations of plates were published at the end of the 18th century by researchers as Euler and Bernoulli. Their research was continued by Tanaka, Chladni, Konig, Rayleigh, Ritz etc.

In the recent times: Timoshenko and Leissa for instance brought important progresses in this domain [7-11], by development of methods in order to solve the plates and establish some solutions of their differential equations of equilibrium.

In the paper, the modal functions for a circular plate clamped all around are derived by using Bessel functions and the normalized modal shapes are illustrated in 3D by using MS Excel.

2. Analytical approach

Following the methodology described in [2] the differential equation of motion for the transverse displacement w of a circular plate is given by:

$$D\nabla^4 w + \rho \frac{\eth^2 w}{\eth t^2} \tag{1}$$

where,

D [Nm] is the flexural rigidity and is defined by:

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{2}$$

E [N/m^2] is Young's modulus

h [m] is the plate thickness

ν is Poisson's ratio

ρ [m] is mass density per unit area of the plate

t [s] is time

$\nabla^4 = (\nabla^2)^2$ and ∇^2 is the Laplacian operator.

Free vibrations are assumed and the motion is expressed as:

$$w = W cos(\omega t) \tag{3}$$

where,

ω [rad/s] is the circular frequency

W is a function only of the position coordinates.

Then, by substituting the equation (3) into equation (1) we obtained:

$$(\nabla^4 - k^4)W = (\nabla^2 + k^2)(\nabla^2 - k^2)W \tag{4}$$

where the dimensionless wave number k defined as:

$$k^4 = \frac{\rho\omega^2}{D} \tag{5}$$

By superimposing the solutions, the complete solution to equation (4) become:

$$\begin{cases} \nabla^2 W_1 + k^2 W_1 = 0 \\ \nabla^2 W_2 - k^2 W_2 = 0 \end{cases} \tag{6}$$

It is assumed that the Fourier components are in θ and the solutions (6) becomes:

$$W(r,\theta) = \sum_{n=0}^{\infty} W_n(r)\cos(n\theta) + \sum_{n=1}^{\infty} W_n{}^*(r)\sin(n\theta) \tag{7}$$

The origin of a polar coordinate system of the circular plate clamped all around is taken to coincide with the center of the circular plate. For the analyzed case, the plate has no internal holes. The symmetry of the boundary conditions respect to one or more diameters of the circular plate exist. In this case the terms involving sin(nθ) are not needed [11].

Taking in consideration the Bessel functions, by substituting the equation (7) into equation (6), the general solution in polar coordinates for the circular plate becomes:

$$W_n(r,\theta) = [A_n J_n(kr) + C_n I_n(kr)]cos(n\theta) \tag{8}$$

where,

n = 0 ... ∞ represents the number of nodal diameters

A_n, B_n are the coefficients obtained from boundary conditions

J_n is the Bessel function of the first kind

I_n is the modified Bessel function of the first kind.

The boundary conditions for a circular plate clamped all around with radius R :

$$\begin{cases} W(R) = 0 \\ \frac{\eth W(R)}{\eth r} = 0 \end{cases} \tag{9}$$

When equation (9) is substituted into equation (8), the existence of a nontrivial solution yields the characteristic determinant:

$$\begin{vmatrix} J_n(\lambda) & I_n(\lambda) \\ J'_n(\lambda) & I'_n(\lambda) \end{vmatrix} = 0 \tag{10}$$

where, λ=kR.

Next, the recursion relations will be used:

$$\begin{cases} \lambda J'_n(\lambda) = nJ_n(\lambda) - \lambda J_{n+1}(\lambda) \\ \lambda I'_n(\lambda) = nI_n(\lambda) + \lambda J_{n+1}(\lambda) \end{cases} \tag{11}$$

The frequency equation is obtained by expanding equation (10):

$$J_n(\lambda) J_{n+1}(\lambda) + I_n(\lambda) J_{n+1}(\lambda) = 0 \tag{12}$$

The solutions of the frequency equation give us the dimensionless wave numbers λ^2, where, n represents the number of nodal diameters, and s the number of nodal circles and does not include the contour circle.

The normalized mode shape function is:

$$\begin{cases} W(r,\theta)_{n,s} = \left[J_n\left(\lambda_{n,s}, \frac{r}{R}\right) - \frac{J_n(\lambda_{n,s})}{I_n(\lambda_{n,s})} I_n\left(\lambda_{n,s}, \frac{r}{R}\right)\right] \cos(n\theta) \\ W(r,\theta)_{n,s} = \left[J_n\left(\lambda_{n,s}, \frac{r}{R}\right) - \frac{J_n(\lambda_{n,s})}{I_n(\lambda_{n,s})} I_n\left(\lambda_{n,s}, \frac{r}{R}\right)\right] \sin(n\theta) \end{cases} \tag{13}$$

3. Results

The dimensionless wave numbers $\lambda_{n,s}^2$ depending on the number of nodal diameters n, respectively the number of nodal circles s are presented in table 1.

Table 1. Dimensionless wave numbers $\lambda_{n,s}^2$

s	Nodal diameters n					
	0	1	2	3	4	5
0	10.21583	21.2604	34.87704	51.03004	69.66583	90.73899
1	39.77115	60.82867	84.58265	111.0214	140.1079	171.8030
2	89.10414	120.0792	153.8151	190.3038	229.5186	271.4282
3	158.1842	199.0534	242.7206	289.1799	338.4112	390.3895
4	247.0064	297.7601	351.3360	407.7295	466.9250	528.9021
5	355.5693	416.2026	479.6751	545.9830	615.1140	687.0511
6	483.8722	554.3824	627.7441	703.9546	783.0036	864.8769

The mode shapes for the circular plate clamped all around is presented in the figures 1 - 7.

Fig. 1 present the first six vibration modes for nodal circle s=0 and nodal diameters n=0, 1, 2, 3, 4 and 5.

Fig. 2 present the first six vibration modes for nodal circle s=1 and nodal diameters n=0, 1, 2, 3, 4 and 5.

Fig. 3 present the first six vibration modes for nodal circle s=2 and nodal diameters n=0, 1, 2, 3, 4 and 5.

Fig. 4 present the first six vibration modes for nodal circle s=3 and nodal diameters n=0, 1, 2, 3, 4 and 5.

Fig. 5 present the first six vibration modes for nodal circle s=4 and nodal diameters n=0, 1, 2, 3, 4 and 5.

Fig. 6 present the first six vibration modes for nodal circle s=5 and nodal diameters n=0, 1, 2, 3, 4 and 5.

Fig. 7 present the first six vibration modes for nodal circle s=6 and nodal diameters n=0, 1, 2, 3, 4 and 5.

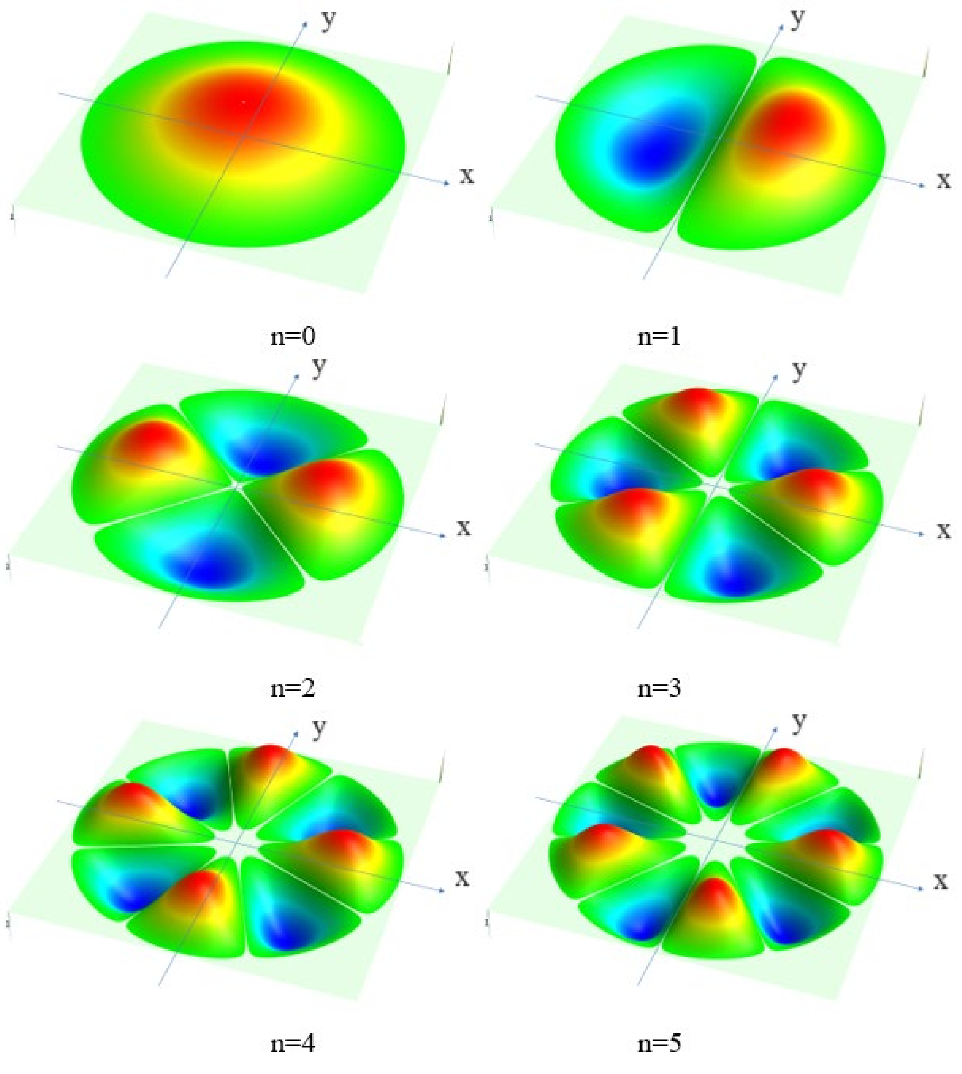


Figure 1. Mode shapes for the circular plate clamped all around; nodal circle s=0, nodal diameters n=0, …, 5.

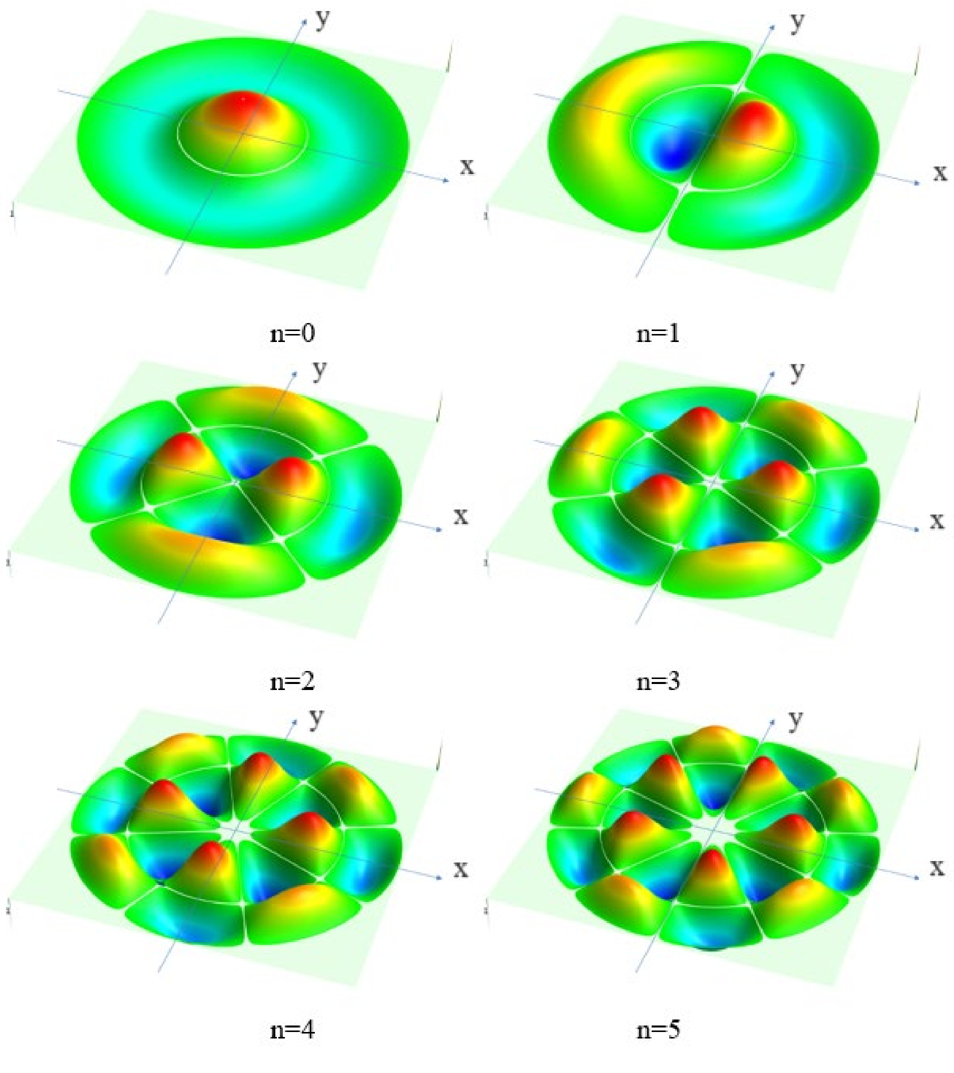


Figure 2. Mode shapes for the circular plate clamped all around; nodal circle s=1, nodal diameters n=0, …, 5.

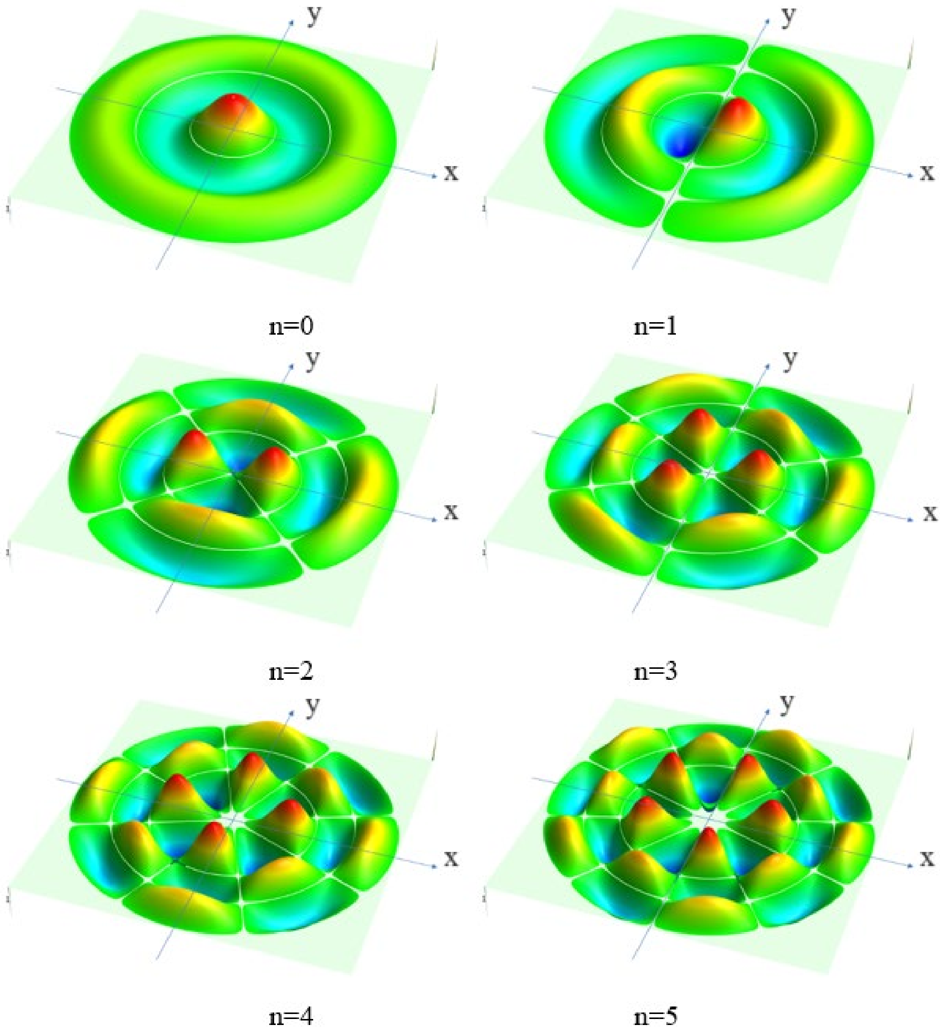


Figure 3. Mode shapes for the circular plate clamped all around; nodal circle s=2, nodal diameters n=0, …, 5.

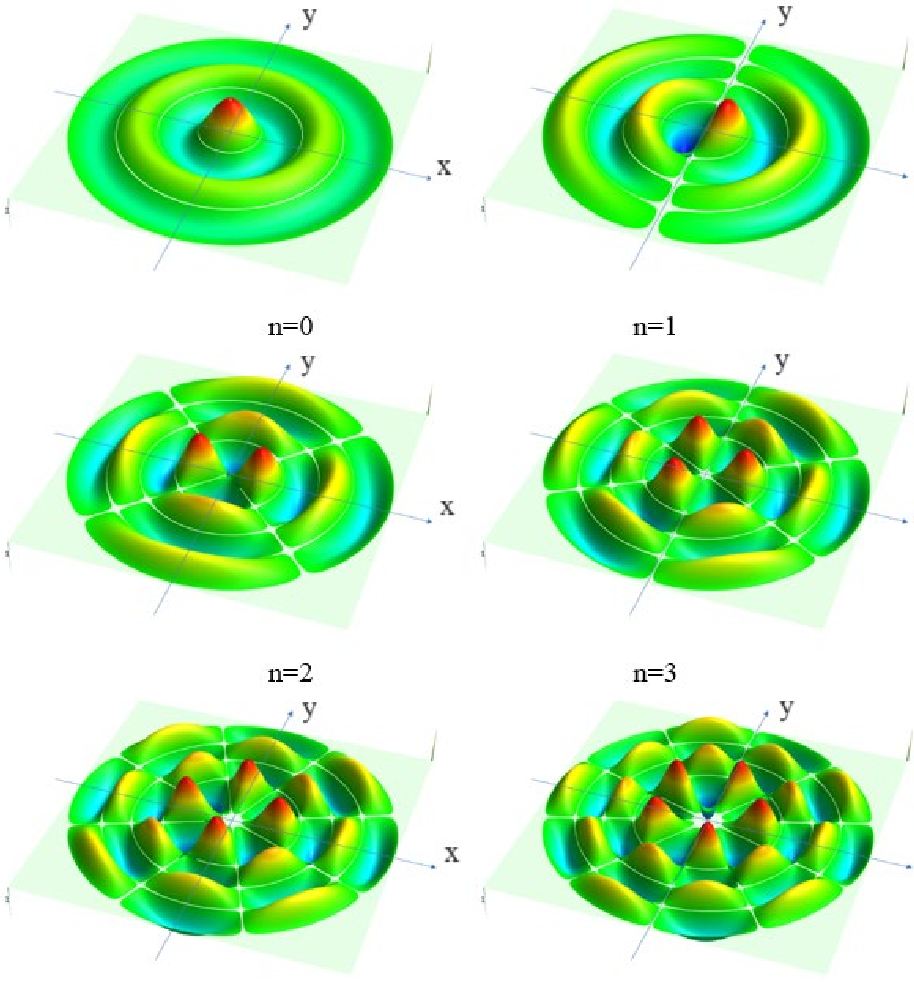


Figure 4. Mode shapes for the circular plate clamped all around; nodal circle s=3, nodal diameters n=0, …, 5.

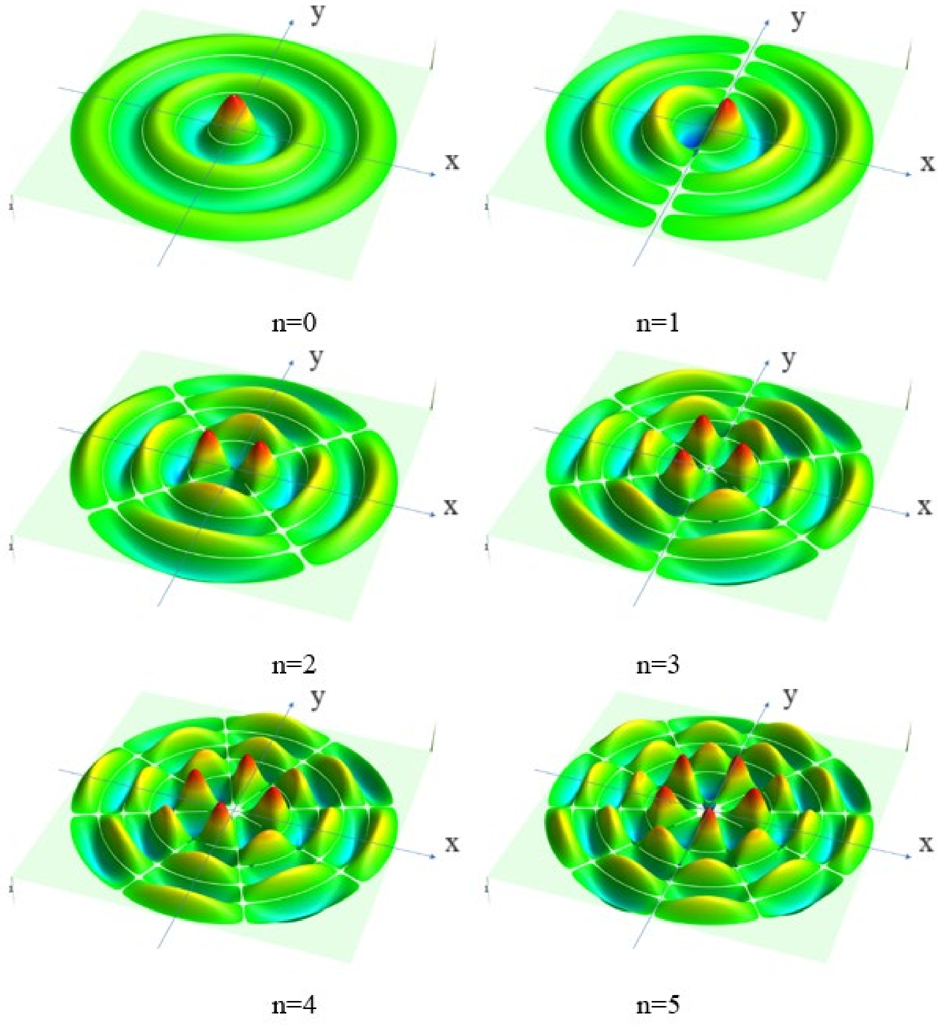


Figure 5. Mode shapes for the circular plate clamped all around; nodal circle s=4, nodal diameters n=0, …, 5.

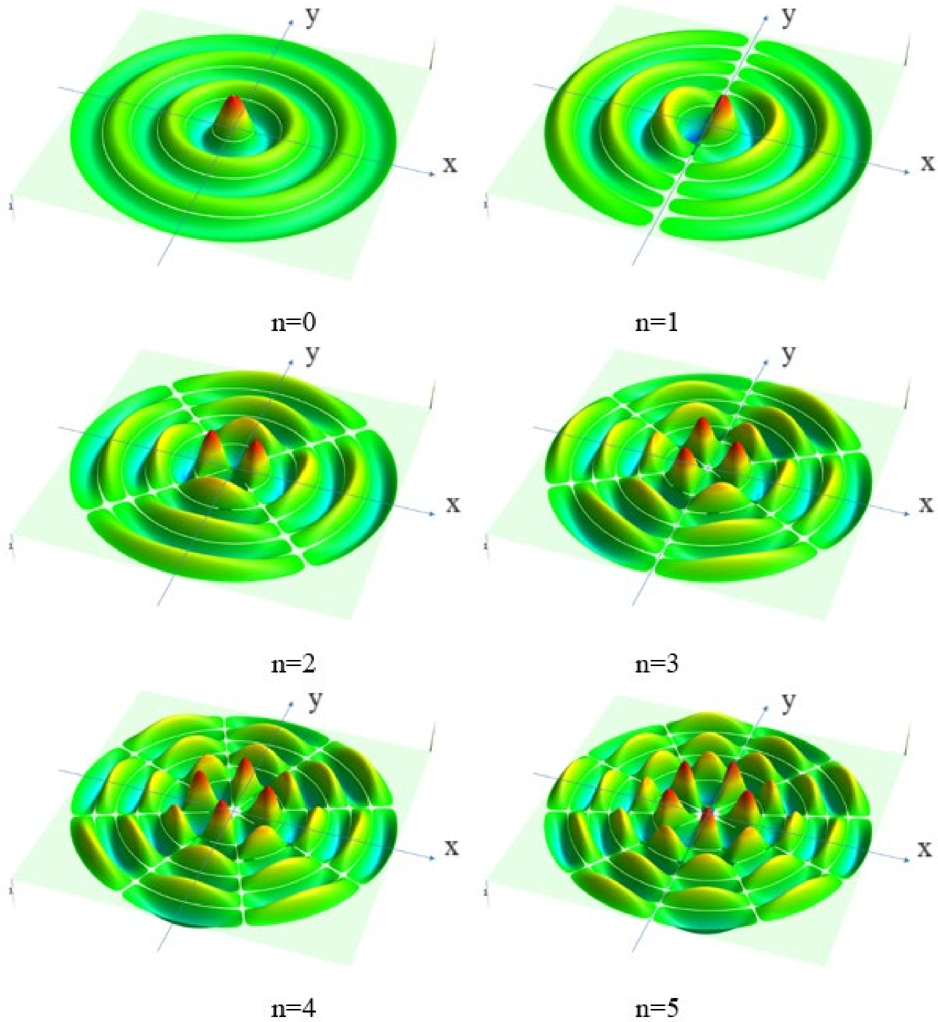


Figure 6. Mode shapes for the circular plate clamped all around; nodal circle s=5, nodal diameters n=0, …, 5.

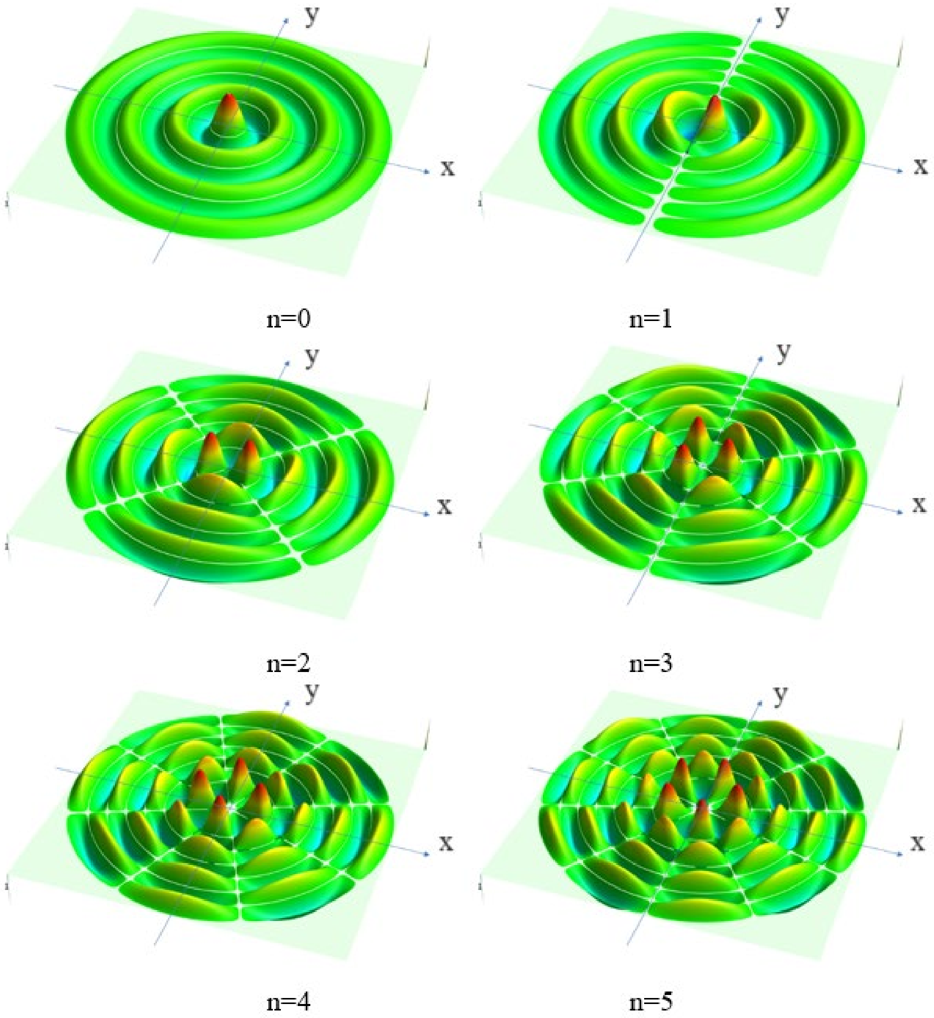


Figure 7. Mode shapes for the circular plate clamped all around; nodal circle s=5, nodal diameters n=0, …, 5.

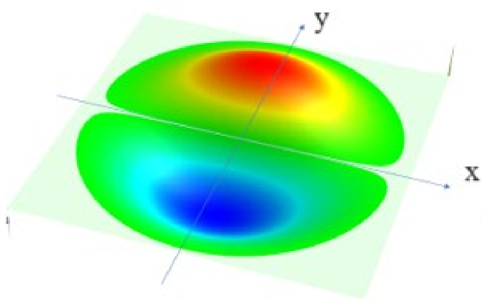
4. Conclusions

The paper presents the vibration modes for a circular plate clamped all around in a 3D representation using MS Excel software. Using the Bessel functions, the frequency equation and the modal function were analytically determined.

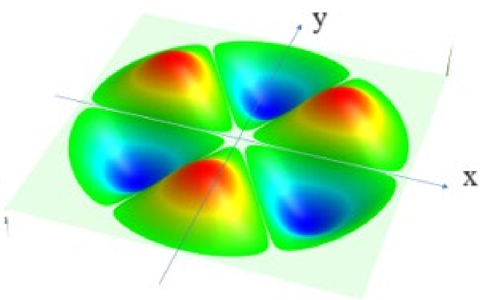
The dimensionless wave numbers $\lambda_{n,s}^2$ were calculated for six values of the nodal diameters n=0, 1, .., 5 and seven values of the nodal circles s=0, 1, ..., 6 and presented in table 1.

For these values of nodal diameters and nodal circles, the modal shapes for a circular plate clamped all around in polar coordinates are illustrated in figures 1 – 7, by using the first relation of the system (13), respectively taking cos(nθ) into account. In this case, the inflection points of the odd nodal diameters (n) pass through the y axis, and the modal function along this direction has zero value.

If the second relation of the system (13) is used, which takes sin(nθ) into account, the representation of the modal shapes is rotated by 90 degrees, respectively, the inflection points of the modal shapes for odd nodal diameters (n) pass through the x axis (fig. 8 – 10).

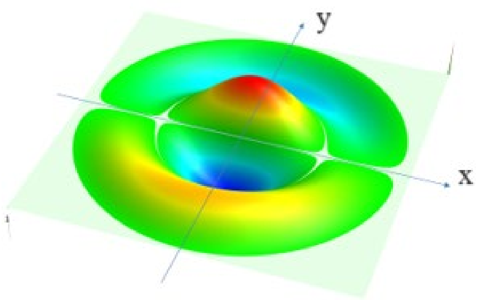


n=1

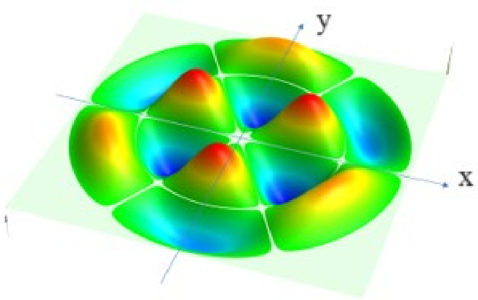


n=3

Figure 8. Mode shapes for the circular plate clamped all around with sin(nθ); nodal circle s=0.



n=1



n=3

Figure 9. Mode shapes for the circular plate clamped all around with sin(nθ); nodal circle s=1.

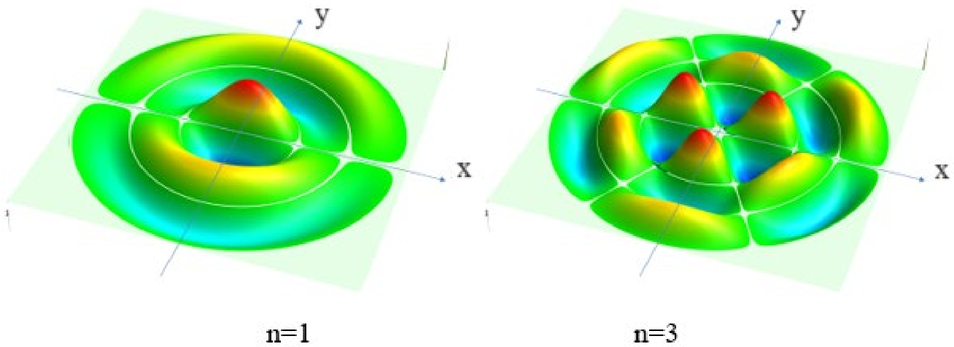


Figure 10. Mode shapes for the circular plate clamped all around with sin(nθ); nodal circle s=2.

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