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Eigenvalues of a continuous beam with two spans

Patric-Timotei Stan, Zeno-Iosif Praisach*

Abstract. *Continuous beams can be found in various engineering applications. The dynamic behavior of continuous beams is strongly influenced by the location of the intermediate support. The paper presents the determination of the frequency equation depending on the location of the intermediate support for a continuous beam with two openings for the case that this support can be placed in any position along the length of the beam. Finally, the obtained curves of the eigenvalues for the first six vibration modes are represented.*

Keywords: *eigenvalues, continuous beam, dynamic behavior*

1. Introduction

The dynamics of structures has as its main purpose the development of methods for determining efforts and deformations in the structure subjected to dynamic actions. A dynamic action is an action whose magnitude, direction or point of application varies over time. Dynamics of structures develops specific calculation methods considering the time variation of the response of a subjected structure to dynamic loads and actions [1 – 5].

Continuous beams are placed on more than two supports, one of these supports is fixed (joint or embedded), and the others are mobile (simple supports). These supports do not prevent the variation of the length in the direction of the axis of the bar, it results that the axial effort does not represent a static indeterminacy of the structures [6 – 8].

Continuous beams are widely used in various engineering applications to effectively support loads. Some engineering applications include bridge structures, building frames, and transportation systems.

The paper presents the determination of the frequency equation depending on the location of the intermediate support for a continuous beam with two openings for the case that this support can be placed in any position along the length of the beam.

2. Euler-Bernoulli theory

To determine the transverse frequency equation for a continuous beam with two openings, the Euler-Bernoulli theory is considered. The continuous beam is a simple beam supported on three supports, and the model of the beam is shown in figure 1.

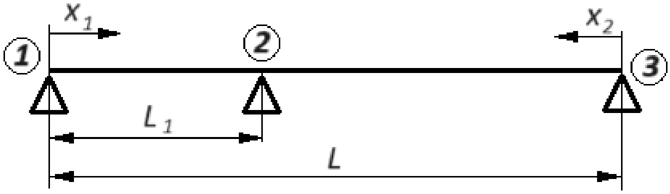

Figure 1. Continuous beam with two spans.

For transverse vibrations, the Euler-Bernoulli model takes into account the bending moment and lateral displacement, and does not take into account the shear deformation and rotary inertia [9 – 10].

The spatial solution of transverse (bending) vibrations, free and undamped, for the Euler-Bernoulli model, has the expression (1):

$$W(x) = Asin(\alpha x) + Bcos(\alpha x) + Csinh(\alpha x) + Dcosh(\alpha x) \tag{1}$$

where,

A, B, C, D are the integration coefficients;

α is the eigenvalue;

W(x) is the modal wave function.

Since the continuous beam has two openings, a modal wave function is considered for each span (fig. 1): 1 – 2, 3 – 2 (2).

$$\begin{cases} W_1(\mathrm{x}_1) = \mathrm{A}_1\sin(\alpha x_1) + \mathrm{B}_1\cos(\alpha x_1) + \mathrm{C}_1\sinh(\alpha x_1) + \mathrm{D}_1\cosh(\alpha x_1) \\ W_2(\mathrm{x}_2) = \mathrm{A}_2\sin(\alpha x_2) + \mathrm{B}_2\cos(\alpha x_2) + \mathrm{C}_2\sinh(\alpha x_2) + \mathrm{D}_2\cosh(\alpha x_2) \end{cases} \tag{2}$$

where,

$x_1 \in (0, L_1)$ is variable along the length of span 1 – 2;

$x_2 \in (0, L - L_1)$ is variable along the length of span 3 – 2;

According to the Euler Bernoulli model, the boundary conditions for each support, taking into account (2) can be written as follows:

$$\begin{cases} W_1(0) = 0 \\ W_1''(0) = 0 \\ W_2(0) = 0 \\ W_2''(0) = 0 \\ W_1(\mathrm{L}_1) = 0 \\ W_2(L - \mathrm{L}_1) = 0 \\ W_1'(\mathrm{L}_1) = -W_2'(\mathrm{L}_2) \\ W_1''(\mathrm{L}_1) = W_2''(\mathrm{L}_2) \end{cases} \tag{3}$$

where,

W_1, W_2 is the deflection;

W_1', W_2' is the slope;

W_1'', W_2'' is directly proportional to bending moment.

$L_1 \in (0, L)$ is the location of the intermediate support.

3. The frequency equation

By solving the system (3) and taking into account (2), the integration constants (4) and the frequency equation (5) are obtained:

$$\begin{cases} B_1 + D_1 = 0 \\ B_1 - D_1 = 0 \Rightarrow \boxed{B_1 = D_1 = 0} \\ B_2 + D_2 = 0 \\ B_2 - D_2 = 0 \Rightarrow \boxed{B_2 = D_2 = 0} \\ A_1 sin(\alpha \mathrm{L}_1) + C_1 sinh(\alpha \mathrm{L}_1) = 0 \Rightarrow \boxed{C_1 = -A_1 \frac{sin(\alpha \mathrm{L}_1)}{sinh(\alpha \mathrm{L}_1)}} \\ A_2 sin\big(\alpha(L - \mathrm{L}_1)\big) + C_2 sinh\big(\alpha(L - \mathrm{L}_1)\big) = 0 \Rightarrow \boxed{C_2 = -A_2 \frac{sin(\alpha(L-\mathrm{L}_1))}{sinh(\alpha(L-\mathrm{L}_1))}} \\ A_1 cos(\alpha \mathrm{L}_1) + C_1 cosh(\alpha \mathrm{L}_1) = -\big[A_2 cos\big(\alpha(L - \mathrm{L}_1)\big) + C_2 cosh\big(\alpha(L - \mathrm{L}_1)\big)\big] \\ -A_1 sin(\alpha \mathrm{L}_1) + C_1 sinh(\alpha \mathrm{L}_1) = -A_2 sin\big(\alpha(L - \mathrm{L}_1)\big) + C_2 sinh\big(\alpha(L - \mathrm{L}_1)\big) \end{cases} \tag{4}$$

By solving the last two equations in system (4) it obtains the integration coefficient A_2 and the frequency equation:

$$\begin{cases} \boxed{A_2 = A_1 \frac{sin(\alpha \mathrm{L}_1)}{sin(\alpha(L-\mathrm{L}_1))}} \\ Z_{11} \cdot Z_{22} + Z_{12} \cdot Z_{21} = 0 \end{cases} \tag{5}$$

where it was denoted,

$$\begin{cases} Z_{11} = cos(\alpha L_1) - \frac{sin(\alpha L_1)}{sinh(\alpha L_1)} cosh(\alpha L_1) \\ Z_{12} = sin(\alpha L_1) \\ Z_{21} = cos\big(\alpha(L - L_1)\big) - \frac{sin\big(\alpha(L-L_1)\big)}{sinh\big(\alpha(L-L_1)\big)} cosh\big(\alpha(L - L_1)\big) \\ Z_{22} = sin\big(\alpha(L - L_1)\big) \end{cases} \tag{6}$$

4. Results

For a normalized length of the continuous beam (L=1) for which the cross-section is constant having the moment of inertia I [m^4] and is made of a material with modulus of elasticity E [N/m^2], the eigenvalues obtained as solution of frequency equation (5), starting from L_1=0.001 till L_1=0.999, with step of 0.001, for the first six vibration mode, are presented in the figures 2 - 7.

Table 1 presents the eigenvalues for the first six vibration modes for L_1=0.001, L_1=0.150, L_1=0.325 and L_1=0.500. Due to the symmetry of the supports, the eigenvalues for L_1 =0.001 are the same with L_1=0.999, for L_1=0.150 are the same with L_1=0.850 and L_1=0.325 are the same with L_1=0.675.

Table 1. Eigenvalues for the first six vibration modes for different values of L_1

L_1	0.001 / 0.999	0.150 / 0.850	0.325 / 0.675	0.500
α_1	3.9292235	4.40911578	5.28238731	6.2831853
α_2	7.07330383	7.98600823	9.41313727	7.85320462
α_3	10.216999	11.5834823	11.26626	12.5663706
α_4	13.3606957	15.1779383	14.6365202	14.1371655
α_5	16.5043945	18.7173101	18.8061283	18.8495559
α_6	19.6480955	21.929824	20.815083	20.4203522

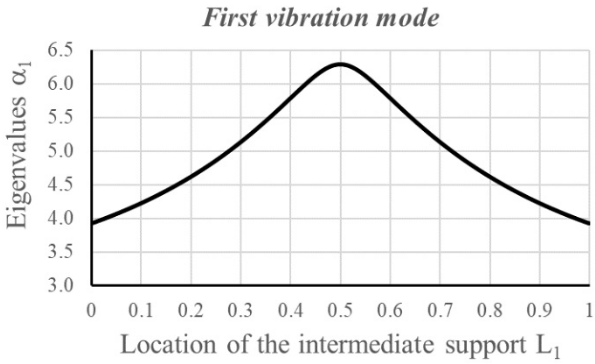

Figure 2. Eigenvalues for the first vibration mode.

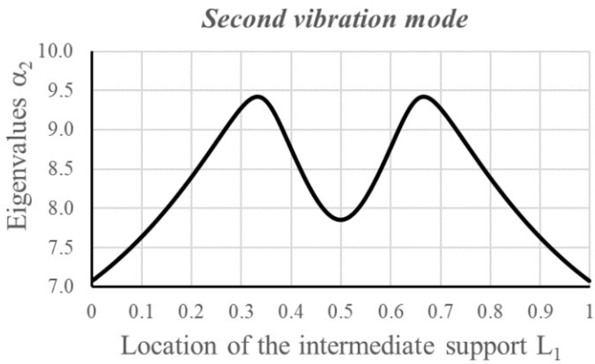

Figure 3. Eigenvalues for the second vibration mode.

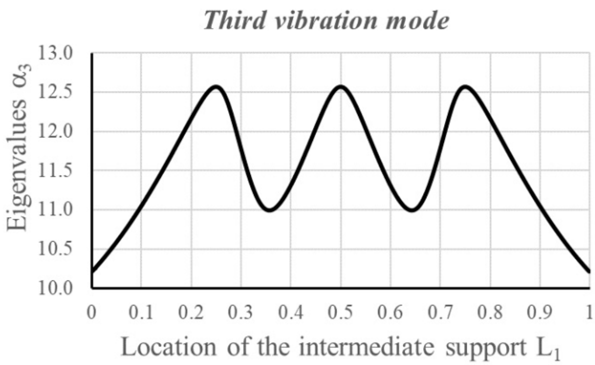

Figure 4. Eigenvalues for the third vibration mode.

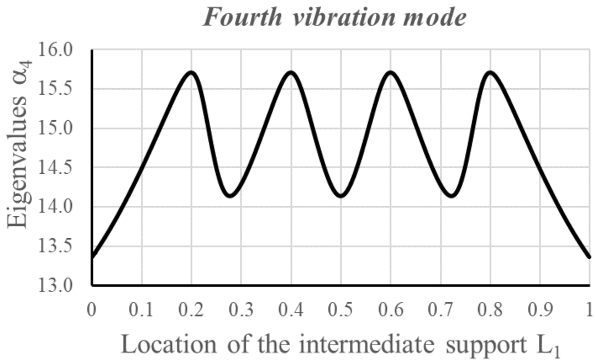

Figure 5. Eigenvalues for the fourth vibration mode.

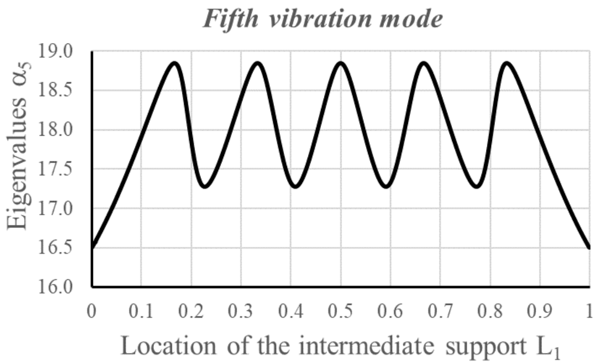

Figure 6. Eigenvalues for the fifth vibration mode.

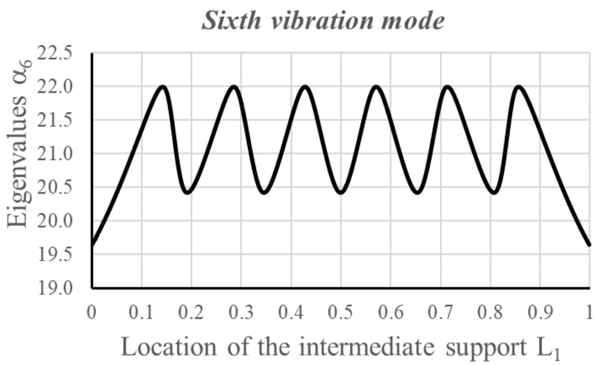

Figure 7. Eigenvalues for the sixth vibration mode.

5. Conclusions

The results obtained for the eigenvalues of the continuous beam with two spans, assuming that the intermediate support can occupy any position along the length L of the beam, is important for calculating the natural frequencies of the beam, considering that the natural frequency of the beam is directly proportional to the square of the eigenvalue.

Knowing the eigenvalues of the continuous beam and the values of the integration coefficients from system (2) we can write the modal function that allows us to draw the modal shapes for the vibration modes of the beam.

From the analysis of figures 2 - 7, it can be seen that the eigenvalues obtained, compared to the middle location of the intermediate support, are symmetrical regardless of the mode of vibration.

When L_1=0.5, the eigenvalues, respectively the natural frequencies, for the odd vibration modes have maximum values. Minimum values are obtained when the intermediate support is positioned very close to the ends of the continuous beam.

As a particularity, the maximum value of the eigenvalues is equal to $(n+1)\cdot\pi$, where n represents the number of the vibration mode.

When the intermediate support is positioned very close to the beam ends, the dynamic behavior of the continuous beam can be associated with a beam clamped at one end and hinged at the other. For these cases, the clamped end is not perfect, it can be considered a weak clamp, because the boundary conditions for the end point are for a hinge end and not a clamp end. In other words, in the hinged end of the continuous beam, the slope does not have zero value as in the case of a clamped end, all the more so as the boundary condition for this hinge imposes a bending moment with zero value.

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Addresses:

- PhD. stud. Eng. Patric-Timotei Stan, Doctoral School of Engineering, Babeș-Bolyai University, Cluj-Napoca, Romania, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa
  patric.stan@ubbcluj.ro
- Lect. PhD. Habil. Zeno-Iosif Praisach, Department of Engineering Science, Faculty of Engineering, Babeș-Bolyai University Cluj-Napoca, Romania, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa
  zeno.praisach@ubbcluj.ro
  (*corresponding author)