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PORTFOLIO OPTIMIZATION ALGORITHMS

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ABSTRACT. A milestone in *Portfolio Theory* is represented by the Mean-Variance Model introduced in 1952 by Harry Markowitz. During the years, mathematicians have developed several different models extending, improving and diversifying the Mean-Variance Model. This paper will briefly present some of these extensions and the resulted models. The aim is to search and identify some connections between portfolio theory and energy production. Analyzing the Mean-Variance Model and its extensions we can conclude that from practical point of view the minimax model is the easiest to be implemented, because the analytical solution is computed with low effort. This model, like all others from Portfolio Theory, has a high sensitivity for mean. We consider that this model fits to our goal (energy optimization) and we intend to implement it in our future research project.

Key words: portfolio, optimization, algorithms

JEL classification: G11

1. Introduction

The optimal allocation of financial resources is of outmost importance. Capital allocation can determine the development or the stagnation of an economic sector, depending on investors' perception and risk aversion.

A milestone in *Portfolio Theory* is represented by the Mean-Variance Model introduced in 1952 by Harry Markowitz (see [21]). The main idea of Markowitz's theory is that portfolio diversification will reduce risk, measured using variance, as a spread of returns around the expected return. The model is formulated in such way that risk (variance) is minimized while expected return does not fall below a predefined level, or maximizes the expected return while risk (variance) does not exceed a predefined level.

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During the years, mathematicians have developed several different models extending, improving and diversifying the Mean-Variance Model. This paper will briefly present some of these extensions and the resulted models. The aim is to search and identify some connections between portfolio theory and energy production. We will insist on models which we consider appropriate to provide some starting points for our goal.

2. Markowitz Model

Portfolio investment problem was first studied by the mathematician Harry Markowitz, which published in March 1952 in *The Journal of Finance* his paper "*Portfolio selection"* (see [21]) considered the mille stone of this field.

In constructing the entire theory of portfolio optimization, Markowitz presumes that an investor will always chose a portfolio which offers a higher profitability for the same risk or a portfolio which offers the same profitability against a lower risk.

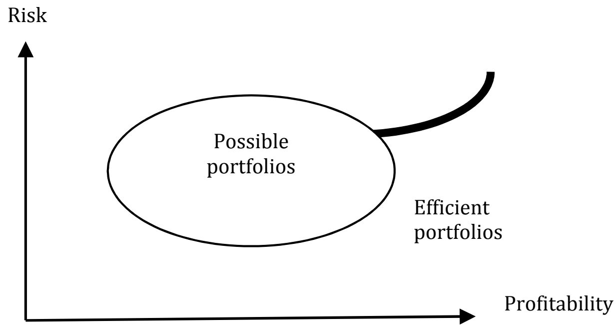


Figure 1: Risk and profitability
Source: Markovitz [21]

Markowitz proved, in a simple way, using tools of analytic geometry, that an efficient method to reduce risk is to diversify the portfolio instead of placing the entire amount in a single asset.

Let's consider a portfolio of n assets denoted by $S_j, j = \overline{1,n}$. An investor, owning the initial amount V_0, focuses the problem to determine which amount to invest in each asset such that the profitability will be maximized and the risk minimized.

Obviously, each considered asset S_j, has a certain rate of return, denoted R_j, which is a random variable. The expected rate of return for asset S_j is $r_j = E[R_j], j = \overline{1,n}$.

Markowitz used variance (L_2 risk function) to measure the risk. The evaluation of risk for the entire portfolio requires that correlation between assets should be considered. If correlation exists, a change in an asset will generate changes for correlated assets. As measure for correlation, the covariance $\sigma_{ij} = E\left[(R_i - r_i)\left(R_j - r_j\right)\right], i, j = \overline{1, n}$ is calculated. It's well known that the covariance between a variable and itself results in variance.

Denoting by x_j the amount invested in asset S_j, $j = \overline{1, n}$, the risk function becomes:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j$$

Let V_0 be the initial wealth of investor and ρ the minimum return expected from the investment. To limit the risk for asset S_j, investor is imposing a maximum amount u_j to be invested.

The mathematical model created by Markowitz (***MV-Mean Variance Model***) (see [21]) is:

$$\begin{cases} min \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\ \sum_{j=1}^{n} r_j x_j \geq \rho V_0 \\ \sum_{j=1}^{n} x_j = V_0 \\ 0 \leq x_j \leq u_j, \qquad j = \overline{1, n} \end{cases}$$

To validate the model, the following conditions have to be fulfilled:

- Random variables R_j, $j = \overline{1, n}$ are normally distributed;
- Investor has a desire to reduce risk.

Although this model created by Markowitz in 1952 is considered to be the foundation of portfolio theory, practitioners are rarely using it. The main drawbacks considered as being the following:

1. Solving the problem is not easy due to quadratic objective function and due to the complexity of covariance matrix, especially when the number of considered assets is large.
2. Investors are considering variance as a non-appropriate measure for risk. Their arguments are that an investor is not satisfied with a small profit or loss; thus he/she is highly satisfied with a high profit, therefore he/she behaves in a different manner toward the risk and reward.

3. The model is very sensitive to errors generated by random variables and their expected value.
4. The model considers a single period portfolio, while almost all portfolios are held over multiple periods or multi-period.

Analyzing the vast literature we have identified five directions that extend, improve and diversify the mean-variance model:

- Extending the model from the single period to multi-period.
- Including the transaction cost in the mathematical model.
- Analyzing the sensitivity for input data.
- Approximation schemes
- Introduction of new risk measures.

In 1972, Merton (see [23]) computed the efficient frontier for Mean-Variance model, using Lagrange multipliers, in a special case when short selling is allowed (no sign restriction on $x_j, j = \overline{1,n}$).

3. Multi-period time horizon approach for Mean-Variance model

By extending the portfolio optimization model to multi-period time horizon, investor has the opportunity to reoptimize the portfolio at some precise time moments. Unfortunately the literature does not offer any indications how the investor should act in the case of reoptimization.

The existing algorithms provide tools for the initial optimization of portfolio; thus the investor could not interfere until the end of the period. In 1967, Smith (see [33]) extended the methodology used for initial configuration of multi-period time horizon portfolios to include intermediate time moments, allowing investor to reoptimize the portfolio during its lifespan.

In 1968, Mossin (see [25]) proves that at each moment of time *t*, $t = \overline{1,T}$ the amount invested in each selected asset depends on the wealth at stage *t*, $t = \overline{1,T}$. Also, he proved that: (i) investment decision for stage *t*, $t = \overline{1,T-1}$ can't be computed before the result of stage *t-1* is known and (ii) decision for stage *t*, $t = \overline{1,T-1}$ considers not only the information regarding returns for stage *t*, but also information regarding returns for stages *t+1, …T.* In the same paper Mossin studied the impact on decision of number of stages until end of the period.

Due to non separability of variance, it is not easy to extend the classical Mean-Variance model for multi-period time horizon. Mathematicians like Merton (see [22]), Samuelson (see [29]) and Fama (see [11]) have developed models which used expected utility as objective function. The expected utility is considered to include besides the wealth at each time stage *t,* the amount used by investor to cover his current costs. They study the relation between investment decisions, consumption decisions and total wealth. We remark that in Portfolio Theory appears the idea of bicriteria problems.

The utility function had overcome some of the difficulties of extending the model to multi-period time horizon, but the use of total or average return still remains a part of the difficulties. In order to overcome this problem, in 1971 Hakkanson (see [14]) developed a model which is using the average compound return.

In 1974, Elton and Gruber (see [9]) realized a very precise inventory of models developed until that moment, specifying for each model the conditions imposed on the objective function, and the probability distribution for rate of return (normal, symmetric stable, log normal, Stable Paretian, none). Using dynamic programming, they developed a multi-period time model which maximizes the utility of an investor which uses at each time stage t, $t = \overline{1,T}$ part of the wealth to cover current costs and the rest to continue the investments. They remark that multi-period time model behaves similar to the single period model. In a different paper (see [10]), also from 1974, Elton and Gruber analyze how geometric mean and expected utility of multi-period returns behave as selection criteria for portfolio. Also, they analyze which is the impact on portfolio performance of the number of reoptimizations made by the investor.

Only in 2000 the analytical solution for multi-period time model was computed. The result is due to Li and Ng (see [20]). They returned to the classical formulation of Markowitz model which maximizes the expected final wealth while variation of total wealth does not exceed a predefined level, or minimizes variation of total wealth, while expected final wealth does not fall below a predefined level.

The mathematical model for the problem is

$$(1)\quad \begin{cases} \max E(V_T) \\ var(V_T) \leq \sigma \\ V_{t+1} = \sum_{j=2}^{n} x_{t,j} r_{t,j} + \left(V_t - \sum_{j=1}^{n} x_{t,j} \right) r_{t,1}, \quad t = \overline{1, T-1} \end{cases}$$

or

$$(2)\quad \begin{cases} \min var(V_T) \\ E(V_T) \geq \varepsilon \\ V_{t+1} = \sum_{j=2}^{n} x_{t,j} r_{t,j} + \left(V_t - \sum_{j=1}^{n} x_{t,j} \right) r_{t,1}, \quad t = \overline{1, T-1} \end{cases}$$

The significance for all notations involved is presented in Section 5.3.

To solve the problem, Li and Ng [20] have used a principle somehow similar to Lagrange multipliers to develop the following problem:

$$(3)\quad \begin{cases} \max\big(E(V_T) - \omega var(V_T)\big) \\ V_{t+1} = \sum_{j=2}^{n} x_{t,j} r_{t,j} + \left(V_t - \sum_{j=1}^{n} x_{t,j} \right) r_{t,1}, \quad t = \overline{1, T-1} \end{cases}$$

where $\omega \in [0, \infty)$.

An equivalent problem for (3) is solved and thus the solution for (3) is computed. Considering the relation between problems (1), (2) and (3), the efficient frontier for multi-period Markowitz model is computed.

4. Transaction cost, sensitivity to input data and approximation schemes

As part of extending the Markowitz model to multi-period time horizon, mathematicians have considered to include transaction costs in the model. Through transaction costs we understand the brokerage fee, the cost incurred by analysis, and any other cost generated in the process of deciding upon placing or not an order, including the price difference generated by the delay in executing an order. Financiers and mathematicians argued that the optimal solution computed with zero transaction costs may be different from the solution when transaction costs occur. In the literature this idea was developed by Constantinides (see [7]), Perold (1988, see [27]), Amihud and Mendelson (1988, see [1]), Dumas and Luciano (see [8]).

Perold [27], respectively Amihud and Mendelson [1] have a more financial oriented approach by analyzing the impact of execution and opportunity cost, respectively of liquidity and marketability of the assets on portfolio construction. Constantinides [7], respectively Dumas and Luciano [8] favored a mathematical approach. They developed multi-period models with infinite horizon and studied the influence of transaction costs on optimal solution. While in Constantinides [7] model the investor is calculating the transaction costs at each time stage as a fixed percentage from the total wealth, in Dumas and Luciano [8] model the transaction costs are consumed only at end moment of time.

The sensitivity of solution to input data was studied by Best and Grauer (see [3] and [4]), Chopra, Hensel and Turner (see [6]). They have studied the sensitivity of solution to changes of mean and/or coefficients in problems restriction. The general conclusion is that an optimal solution is extremely sensitive to mean and some adjustments of input data may improve the solution.

Approximation schemes were developed to overcome difficulties (computing the covariance matrix for example) created by specific form of portfolio optimization problem. Contributions to this direction are due to mathematicians like Sharpe (see [30], see [31], see [32]), Stone (1973, see [34]), van Hohenbalken (1975, see [35]).

Sharpe (see [30]) created an index model. He introduced and computed an index for all assets evaluated for investment and instead of calculating the correlations for all pairs of assets, the correlation between each asset and the index is calculated, reducing the time allocated for the operation. More recent, the index models have been further developed by Lee, Finnerty and Wort (see [19]), Huang and Qiao (see [15])

In 1971, Sharpe (1971, see [17]) remarked that "*if the essence of the portfolio analysis problem could be adequately captured in a form suitable for linear programming methods, the prospect for practical application would be greatly enhanced.*" This remark opened the way of linear approximation of Mean-Variance Model on which also Stone (1973, see []) has contributed.

Van Hohenbalken (see [35]) created a model which makes successive approximations of the constraint set.

5. Introduction of new risk measures

Using variance to evaluate the risk creates serious difficulties both in computing the covariance matrix and extending the single period Mean-Variance model to multi-period time horizon. It is well known that variance is a square function. Analyzing the graph of x^2 and $|x|$ functions reveals a similarity, which led mathematicians to develop new measures for risk.

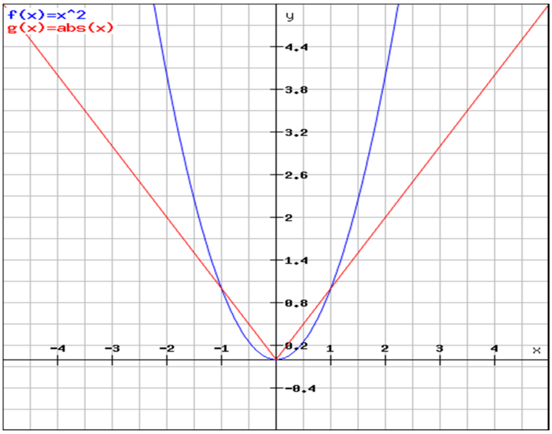


Figure 2: x^2 and $|x|$ functions

Source: author's own work

5.1. Mean absolute deviation model (MAD)

In 1988, Hiroshi Konno (see [17]) proposed to use absolute deviation as a measure for risk (L_1 risk function). The risk for a portfolio of *n* assets is:

$$\omega = E\left[\left|\sum_{j=1}^{n} R_j x_j - \sum_{j=1}^{n} r_j x_j\right|\right]$$

If R_j is normally distributed, then the two measures for risk (measure of Markowitz and measure of Konno) are equivalent.

The model created by Konno and Yamazaki (see [18]) in order to optimize the investment is:

$$\begin{cases} minE\left[\left|\sum_{j=1}^{n} R_j x_j - \sum_{j=1}^{n} r_j x_j\right|\right] \\ \sum_{j=1}^{n} r_j x_j \geq \rho V_0 \\ \sum_{j=1}^{n} x_j = V_0 \\ 0 \leq x_j \leq u_j, \qquad j = \overline{1, n} \end{cases}$$

Being a linear programming problem, this model eliminates the difficulty of solving the problem which appeared to the Markowitz model.

5.2. Minimax model for single period portfolio selection

Generally, the risk is perceived as a variation between total profitability and total expected profitability. Markowitz used variance to evaluate this variation, which means that

$$\omega = E\left[\left|\sum_{j=1}^{n} R_j x_j - \sum_{j=1}^{n} r_j x_j\right|\right]^2.$$

Instead of calculating the expected value for this deviation, it might be considered the probability that the deviation is greater than a predefined value. This means

$$P\left(\left|\sum_{j=1}^{n} R_j x_j - \sum_{j=1}^{n} r_j x_j\right| \geq \varepsilon\right).$$

To keep the variance as low as possible, it's enough to keep the above probability as low as possible.

Using the Markov inequality $P(X \geq a) \leq \frac{1}{a} E(X)$ we obtain:

$$P\left(\left|\sum_{j=1}^{n} R_j x_j - \sum_{j=1}^{n} r_j x_j\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon} E\left(\left|\sum_{j=1}^{n} R_j x_j - \sum_{j=1}^{n} r_j x_j\right|\right) \leq \frac{1}{\varepsilon} E\left(\left|\sum_{j=1}^{n} R_j - \sum_{j=1}^{n} r_j\right| x_j\right) \leq \frac{1}{\varepsilon} \sum_{j=1}^{n} E(|R_j x_j - r_j x_j|) \leq \frac{n \max\limits_{1 \leq j \leq n} E(|R_j x_j - r_j x_j|)}{\varepsilon}$$

Using this argument, Cai (see [5]) introduced a new measure for risk:

$$\omega_\infty = \max_{1 \leq j \leq n} E(|R_j x_j - r_j x_j|)$$

Which is the meaning for this risk proposed by Cai? For each individual asset, the absolute deviation between profitability and expected profitability is calculated and the maximum of all these values is the portfolio risk.

Which is the link between risk defined by Markowitz (ω) and risk defined by Cai (ω_∞)? If ω_∞ is small then also the variance (ω) is small. If the variance (ω) is small, there is not a guarantee that ω_∞ is small.

Using this risk measure, Cai introduced a new portfolio optimization model, known as ***Minimax model*** (see [5]) and formulated as:

$$(PO\ 1)\qquad \begin{cases} min\left(\max\limits_{1\le j\le n} E(|R_j x_j - r_j x_j|)\,; -\sum\limits_{j=1}^{n} r_j x_j\right) \\ \sum\limits_{j=1}^{n} x_j = V_0 \\ x_j \ge 0, \qquad\qquad j=\overline{1,n} \end{cases}$$

It's obviously that $(PO\ 1)$ is a bicriteria optimization problem.

Denoting by $q_j = E(|R_j - r_j|), j=\overline{1,n}$ the expected absolute deviation for R_j, risk becomes:

$$\omega_\infty = \max_{1\le j\le n} q_j x_j$$

Solving this problem means to determine the amount to be invested in each asset, as such that the total wealth is maximized and the investment risk is minimized. From mathematical point of view, the solution is an efficient point, defined as:

An admissible solution x is efficient if there does not exist an admissible solution y such that:

$$\max_{1\le j\le n} q_j y_j \le \max_{1\le j\le n} q_j x_j$$
$$\sum_{j=1}^{n} r_j x_j \le \sum_{j=1}^{n} r_j y_j$$

Formulated in natural language, it is not possible to have another allocation with a smaller risk and a higher profitability.

To simplify the bicriteria problem, it can be rewritten it as:

$$(POB\ 1)\qquad \begin{cases} min\left(y; -\sum\limits_{j=1}^{n} r_j x_j\right) \\ q_j x_j \le y, \qquad j=\overline{1,n} \\ \sum\limits_{j=1}^{n} x_j = V_0 \\ x_j \ge 0, \qquad j=\overline{1,n} \end{cases}$$

Referring to a result of Yu, from 1974, which links bicriteria and parametric problems, $(POB\ 1)$ is rewritten as:

$$(PO(\lambda)\ 1) \qquad \begin{cases} min\left[\lambda y - (1-\lambda)\sum_{j=1}^{n} r_j x_j\right] & \\ q_j x_j \le y, & j=\overline{1,n} \\ \sum_{j=1}^{n} x_j = V_0 & \\ x_j \ge 0, & j=\overline{1,n} \end{cases}$$

Proposition 1 (Yu, 1974) (see [41] and [5])

(x,y) is an efficient solution for $(POB\ 1)$ *if and only if there exist* $\lambda \in (0,1)$ *such that* (x,y) *is optimal solution for* $(PO(\lambda)\ 1)$.

λ is the investor's acceptance for risk. The investor will accept a higher risk when λ is closer to 1.

Considering the equivalence between $(PO\ 1)$ and $(POB\ 1)$, the result of Yu is extended between $(PO\ 1)$ and $(PO(\lambda)\ 1)$. For a fixed λ, the optimal solution of $(PO(\lambda)\ 1)$ is the efficient solution for $(PO\ 1)$. An efficient frontier will be obtained by solving the problem $(PO(\lambda)\ 1)$ for any $\lambda \in (0,1)$ which means that, for any risk tolerance, the investor will determine the amount to be invested in each asset.

The ***Minimax Model*** created by Cai has the advantage of providing an analytical solution, which makes the model easy to be utilized by practitioners. Moreover, the model allows the investor to avoid the calculation of covariance, which requires an important effort for large portfolios. Unfortunately, Minimax Model does not allow short-selling, which facilitates solution in Mean Variance Model. The short-selling impact on the solution is not known. Also the impact of bounding the amount invested in each asset on the solution is not known.

Solving the $(PO(\lambda)\ 1)$ problem is a 2 step process.

First, all assets are ranked by the expected rate of return and then the selected assets for the investment are chosen.

Second, the amount invested in each asset is computed. To compute this amount, the risk for each asset is evaluated and the invested amount is computed such that the exposure is similar for all assets. Through exposure we understand $q_j x_j, j=\overline{1,n}$.

For a fixed λ the solution for problem $(PO(\lambda)\ 1)$ is computed. The following theorem provides the solution.

Theorem 2 (solution for parametric problem) (see [5])

For $\forall\lambda \in (0,1)$ *the solution for parametric problem* $(PO(\lambda)\ 1)$ *is:*

$$(1)\qquad x_j^* = \begin{cases} \dfrac{V_0}{q_j}\left(\displaystyle\sum_{l\in\mathcal{F}} \frac{1}{q_l}\right)^{-1}, & j \in \mathcal{F} \\ 0, & j \notin \mathcal{F} \end{cases}$$

$$(2)\qquad y^* = V_0\left(\sum_{l\in\mathcal{F}} \frac{1}{q_l}\right)^{-1}$$

where $\mathcal{F}$ *is the set of assets to be invested and is computed as following:*

If $\exists k = \overline{0, n-2}$ *such that:*

$$(3)\qquad \frac{r_n - r_{n-1}}{q_n} < \frac{\lambda}{1-\lambda}$$

$$(4)\qquad \frac{r_n - r_{n-2}}{q_n} + \frac{r_{n-1} - r_{n-2}}{q_{n-1}} < \frac{\lambda}{1-\lambda}$$

…………………

$$(5)\qquad \frac{r_n - r_{n-k}}{q_n} + \frac{r_{n-1} - r_{n-k}}{q_{n-1}} + \cdots + \frac{r_{n-k+1} - r_{n-k}}{q_{n-k+1}} < \frac{\lambda}{1-\lambda}$$

$$(6)\qquad \frac{r_n - r_{n-k-1}}{q_n} + \frac{r_{n-1} - r_{n-k-1}}{q_{n-1}} + \cdots + \frac{r_{n-k} - r_{n-k-1}}{q_{n-k}} \geq \frac{\lambda}{1-\lambda}$$

then $\mathcal{F} = \{n, n-1, \ldots, n-k\}$;
else $\mathcal{F} = \{n, n-1, \ldots, 1\}$.

This is the meaning for the relation

$$\frac{r_n - r_{n-1}}{q_n} < \frac{\lambda}{1-\lambda}$$

Rewriting the above inequality as $(1-\lambda)(r_n - r_{n-1}) < \lambda q_n$ we deduce that, if the inequality holds then the return rate for asset S_n is big enough to make the asset attractive.

Theorem 2 gives the optimal solution when all assets are risky. Risk free assets can not be neglected (for example government securities) but they small rates of return. Without loss of generality, risk free asset will be on position 1 in the ranking process. To determine whether the risk free asset is selected or not for investment, the following inequality has to be checked

$$\frac{r_n - r_1}{q_n} + \frac{r_{n-1} - r_1}{q_{n-1}} + \cdots + \frac{r_2 - r_1}{q_2} < \frac{\lambda}{1-\lambda}$$

If the inequality holds, the entire amount will be invested in the risk free asset.

Having the optimal solution for parametric problem ($PO(\lambda)$ 1), efficient solution for portfolio optimization problem (PO 1) has to be computed.

Denoting by

$$\beta_k = \frac{r_n - r_{n-k}}{q_n} + \frac{r_{n-1} - r_{n-k}}{q_{n-1}} + \cdots + \frac{r_{n-k+1} - r_{n-k}}{q_{n-k+1}}, \quad k = \overline{1, n-1}$$

$$\beta_0 = 0$$

inequalities *(3), (4), (5)* and *(6)* will be

$$\beta_1 < \frac{\lambda}{1-\lambda}, \qquad \beta_2 < \frac{\lambda}{1-\lambda}, \qquad \ldots\ldots\ldots \quad \beta_k < \frac{\lambda}{1-\lambda}, \qquad \beta_{k+1} \geq \frac{\lambda}{1-\lambda}$$

Obviously $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_{n-2} \leq \beta_{n-1}$

To compute the set $\mathcal{F}$ of assets to be invested, *k* has to be determined such that:

$$\beta_k < \frac{\lambda}{1-\lambda}, \qquad \beta_{k+1} \geq \frac{\lambda}{1-\lambda}$$

Solving the above inequalities we have $\frac{\beta_k}{1+\beta_k} < \lambda \leq \frac{\beta_{k+1}}{1+\beta_{k+1}}$ and denoting by $\underline{\lambda}_k = \frac{\beta_k}{1+\beta_k}$, $\overline{\lambda}_k = \frac{\beta_{k+1}}{1+\beta_{k+1}}$, we get $\lambda \epsilon (\underline{\lambda}_k, \overline{\lambda}_k]$.

According to *Theorem 2* the set of assets to be invested is $\mathcal{F} = \{n, n-1, \ldots, n-k\}$ where $k = \overline{0, n-2}$ and the invested amount and risk will be computed using *(1)* and *(2).* This means that for each λ in $(\underline{\lambda}_0, \overline{\lambda}_0]$, $(\underline{\lambda}_1, \overline{\lambda}_1]$, …. $(\underline{\lambda}_{n-2}, \overline{\lambda}_{n-2}]$ solution for parametric problem ($PO(\lambda)$ 1) is computed. For interval $(\underline{\lambda}_{n-1}, \overline{\lambda}_{n-1}]$ solution coresponds to case $\mathcal{F} = \{n, n-1, \ldots, 1\}$ from *Theorem 2.*

Has the parametric problem ($PO(\lambda)$ 1) a unique solution? Cai proved that solution is unique on intervals$(\underline{\lambda}_k, \overline{\lambda}_k)$, while for $\lambda = \overline{\lambda}_k$ problem has multiple solution, which is obtained by considering for investment the first asset which was eliminated in the ranking process. The amount invested in this asset will come from reducing the amount invested in the other considered assets. Diversifying the portfolio will determine a risk decrease and also a total wealth decrease. Risk decrease will be Δ_y, where

$$0 \leq \Delta_y \leq \frac{y^*}{1+\sum_{l\in\mathcal{F}} \frac{q_{n-k-1}}{q_l}}$$

The value chosen by investor for Δ_y will have a huge impact on final solution.

Following theorem gives the efficient frontier for portfolio optimization problem $(PO\ 1)$.

Theorem 3 (solution for portfolio optimization problem)(see [5])

Efficient frontier for portfolio optimization problem $(PO\ 1)$ *is computed by considering* n *intervals* $(\underline{\lambda}_k, \overline{\lambda}_k)$, $k=\overline{0,n-1}$ *with* $\overline{\lambda}_k = \underline{\lambda}_{k+1}$, $k=\overline{0,n-2}$, *for which the following holds:*

1. *For each* $k=\overline{0,n-1}$, *problem* $(PO\ 1)$ *has a unique solution on interval* $(\underline{\lambda}_k, \overline{\lambda}_k)$.

   The amount invested in each asset is:

$$x_j^* = \begin{cases} \frac{V_0}{q_j}\left(\sum_{l\in\mathcal{F}} \frac{1}{q_l}\right)^{-1}, & j\in\mathcal{F} \\ 0, & j\notin\mathcal{F} \end{cases}$$

where $\mathcal{F}=\{n, n-1, \dots, n-k\}$.

Total portfolio risk is:

$$y^* = V_0\left(\sum_{l\in\mathcal{F}} \frac{1}{q_l}\right)^{-1}$$

Total wealth is:

$$z^* = V_0 \sum_{j\in\mathcal{F}} \frac{r_j}{q_j}\left(\sum_{l\in\mathcal{F}} \frac{1}{q_l}\right)^{-1}$$

2. *For each* $k = \overline{0, n-2}$ *and* $for\ \lambda = \bar{\lambda}_k = \underline{\lambda}_{k+1}$ *problem* $(PO\ 1)$ *has multiple solutions.*

   The amount invested in each asset is:

$$x_j^0 = \begin{cases} x_j^* - \Delta_j, & j\epsilon\mathcal{F} \\ \Delta_l, & l = n-k-1 \\ 0, & else \end{cases}$$

   where $\mathcal{F} = \{n, n-1, \dots, n-k\}$.

   Total portfolio risk is:

$$y^0 = y^* - \Delta_y$$

   Total wealth is:

$$z^0 = z^* - \Delta_y \frac{\bar{\lambda}_k}{1-\bar{\lambda}_k}$$

   Where:

$$0 \le \Delta_y \le \frac{y^*}{1 + \sum_{j\in\mathcal{F}} \frac{q_{n-k-1}}{q_j}}$$

$$\Delta_j = \frac{\Delta_y}{q_j}, j\epsilon\mathcal{F}$$

$$\Delta_l = \sum_{j\in\mathcal{F}} \Delta_j$$

If the portfolio contains also a risk free asset, then the solution is presented by the following theorem.

Theorem 4 (solution for portfolio optimization problem in case of risk free asset) (see [5])

The efficient frontier for portfolio optimization problem $(PO\ 1)$ *is computed by considering n-i*$_0$*+1 interval* $(\underline{\lambda}_k, \bar{\lambda}_k)$, $k = \overline{0, n-\iota_0-1}$ *with* $\bar{\lambda}_k = \underline{\lambda}_{k+1}$, $k = \overline{0, n-\iota_0-1}$ *for which the following are true:*

1. *For* $k = \overline{0, n-\iota_0-1}$, *problem* $(PO\ 1)$ *has unique solution on intervals* $(\underline{\lambda}_k, \bar{\lambda}_k)$.

The amount invested in each asset is

$$x_j^* = \begin{cases} \frac{V_0}{q_j}\left(\sum_{l\in\mathcal{F}}\frac{1}{q_l}\right)^{-1}, & j\in\mathcal{F} \\ 0, & j\notin\mathcal{F} \end{cases}$$

with $\mathcal{F} = \{n, n-1, \dots, n-k\}$.

Total risk of portfolio is

$$y^* = V_0\left(\sum_{l\in\mathcal{F}}\frac{1}{q_l}\right)^{-1}$$

and total wealth is

$$z^* = V_0\sum_{j\in\mathcal{F}}\frac{r_j}{q_j}\left(\sum_{l\in\mathcal{F}}\frac{1}{q_l}\right)^{-1}$$

2. *For each* $k = \overline{0, n-\iota_0-1}$ *and for* $\lambda = \bar{\lambda}_k = \underline{\lambda}_{k+1}$ *problem* $(PO\ 1)$ *has multiple solutions.*

   The amount invested in each asset is

$$x_j^0 = \begin{cases} x_j^* - \Delta_j, & j\in\mathcal{F} \\ \Delta_l, & l = n-k-1 \\ 0, & else \end{cases}$$

with $\mathcal{F} = \{n, n-1, \dots, n-k\}$.

Total risk of portfolio is

$$y^0 = y^* - \Delta_y$$

and total wealth is

$$z^0 = z^* - \Delta_y\frac{\bar{\lambda}_k}{1-\bar{\lambda}_k}$$

where

$$0 \le \Delta_y \le \frac{y^*}{1+\sum_{j\in\mathcal{F}}\frac{q_{n-k-1}}{q_j}}$$

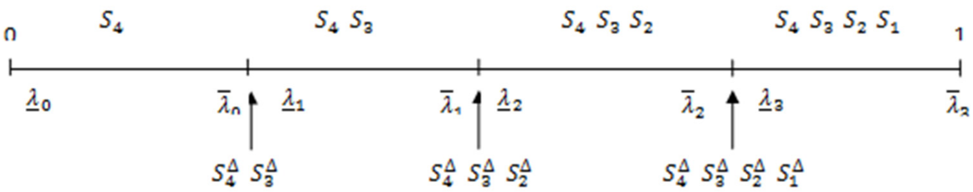
$$\Delta_j = \frac{\Delta_y}{q_j}, j\in\mathcal{F}$$

$$\Delta_l = \sum_{j\in\mathcal{F}}\Delta_j$$

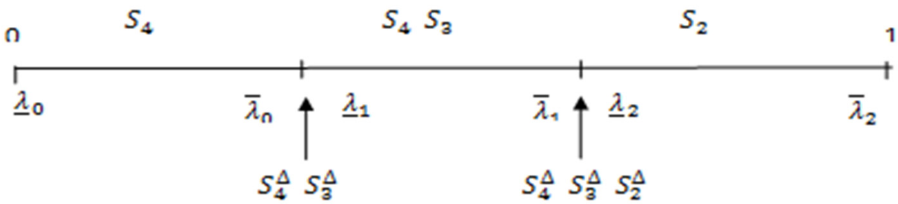
3. *For interval* $(\underline{\lambda}_{n-i_0}, 1)$ *problem* $(PO\ 1)$ *has unique solution.*
The entire amount is invested in riskless asset, so $x_{i_0} = V_0$.
Total portfolio risk is y=0.
Total wealth is $z = r_{i_0}V_0$.

The following diagram presents an optimization process for a portfolio of 4 assets.

Portfolio of 4 risky assets



Portfolio of 4 assets, with 2nd asset being risk free



5.3. Minimax models for multi-period portfolio selection

Yu model

The first analytical solution for a multi-period portfolio problem was obtained by Li and Ng (see [20]). The problem, which actually is an extension of Markowitz Model from single period to multi-period, was briefly presented in Section 3.

A second analytical solution was obtained by Yu, Wang, Lai and Chao in 2005 (see [40]).

The main difference between those two models is given by the way the mathematical model is developed.

In Li and Ng model, the investor seeks either to maximize the total wealth while risk is less than or equal to a defined level or to minimize the risk while the total wealth does not decrease under a defined level.

In the model of Yu, investor seeks to minimize the risk and to maximize the total wealth.

The two groups of researchers are defining risk in different ways. Li and Ng are defining risk as variance of total wealth, while Yu is defining risk as a sum over all periods for maximum of absolute deviation calculated for each individual asset.

Analyzing the two different approaches it comes out that Yu's model is more conservative, due to the fact that it does not allow short-selling.

An investor which owns an initial wealth V_0 will invest it in a portfolio of assets $S_1, S_2 \dots S_n$ for a time horizon *1,2,...., T.* Knowing that investor may step in and reallocate the amount between assets only at the moments *1, 2, ... t, ... T-1*, he has to compute the amount $x_{tj}, t = \overline{1,T}, j = \overline{1,n}$ allocated to each asset $Sj, j = \overline{1,n}$ such that at time period T the total wealth will be maximized and total risk will be minimized.

Each asset S_j, $j = \overline{1,n}$ has at time moment *t*, $t = \overline{1,T}$ a certain rate of return, denoted by R_{tj}. Expected value for the random variable R_{tj}, is r_{tj}. At each moment of time investor will not introduce and will not extract money from the system, so the following holds:

$$V_{t-1} = \sum_{j=1}^{n} x_{tj}, \qquad t = \overline{1,T}$$

By $R_t = (R_{t1}, R_{t2}, \dots, R_{tn})$ is denoted the vector of rates of return at time moment *t*, and $x_t = (x, x_{t2}, \dots, x_{tn})$ is the vector of the amount invested in each asset.

At the end of each time period, the investor is computing the total wealth according to

$$V_t = V_{t-1} + R_t x_t, t = \overline{1,T} \, .$$

The investment risk at each time moment *t*, $t = \overline{1,T}$ is computed according to

$$\omega_t(x_t) = \max_{1 \le j \le n} E\left(\left|R_{tj}x_{tj} - r_{tj}x_{tj}\right|\right)$$

and the total risk for the investment is

$$\omega_t = \omega_{t-1} + \max_{1 \le j \le n} E\left(\left|R_{tj}x_{tj} - r_{tj}x_{tj}\right|\right)$$

Using the above notations, the mathematical model will be:

$$\text{(PO 2)}\quad \begin{cases} min\left(\omega_T, -E(V_T)\right) \\ V_t = V_{t-1} + R_t x_t, & t = \overline{1,T} \\ \omega_t = \omega_{t-1} + \max\limits_{1\le j\le n} E\left(\left|R_{tj}x_{tj} - r_{tj}x_{tj}\right|\right), & t = \overline{1,T} \\ V_{t-1} = \sum\limits_{j=1}^{n} x_{tj}\,, & t = \overline{1,T} \\ x_{tj} \ge 0, & t = \overline{1,T}, j = \overline{1,n} \end{cases}$$

Analyzing the objective function, is clear that we have a bicriteria optimization problem. In order to simplify (PO 2) we introduce the following constraint

$$E\left(\left|R_{tj}x_{tj} - r_{tj}x_{tj}\right|\right) \le z_t, j = \overline{1,n} \text{ and } t = \overline{1,T}$$

this conducts to the following equivalent problem

$$\text{(POB 2)}\quad \begin{cases} min\left(\omega_T, -E(V_T)\right) \\ V_t = V_{t-1} + R_t x_t, & t = \overline{1,T} \\ \omega_t = \omega_{t-1} + z_t, & t = \overline{1,T} \\ E\left(\left|R_{tj}x_{tj} - r_{tj}x_{tj}\right|\right) \le z_t, & j = \overline{1,n}, t = \overline{1,T} \\ V_{t-1} = \sum\limits_{j=1}^{n} x_{tj}\,, & t = \overline{1,T} \\ x_{tj} \ge 0, & t = \overline{1,T}, j = \overline{1,n} \end{cases}$$

The equivalence between (PO 2) and (POB 2) is sustained by following proposition.

Proposition 5 (see [40]).

If (x,z) is an efficient solution for (POB 2)*, then x is the efficient solution for* (PO 2)*.*

If x is the efficient solution for (PO 2)*, then (x,z) with* $z = (z_1, z_2, \dots, z_T)$ *and* $z_t = \max\limits_{1\le j\le n} E\left(\left|R_{tj}x_{tj} - r_{tj}x_{tj}\right|\right)$*,* $t = \overline{1,T}$ *is the efficient solution for* (POB 2)*.*

To solve the bicriteria problem (POB 2) the result of Yu (*Proposition 1)* is employed and (POB 2) is converted in a parametric optimization problem.

For any $\lambda \in (0,1)$ the following problem is obtained:

$$(PO(\lambda)\ 2) \qquad \begin{cases} min\big(\lambda\omega_T - (1-\lambda)E(V_T)\big) & \\ V_t = V_{t-1} + R_t x_t, & t = \overline{1,T} \\ \omega_t = \omega_{t-1} + z_t, & t = \overline{1,T} \\ E\big(\left|R_{tj}x_{tj} - r_{tj}x_{tj}\right|\big) \le z_t, & j = \overline{1,n}, t = \overline{1,T} \\ V_{t-1} = \sum_{j=1}^{n} x_{tj}\,, & t = \overline{1,T} \\ x_{tj} \ge 0, & t = \overline{1,T}, j = \overline{1,n} \end{cases}$$

The solution for $(PO(\lambda)\ 2)$ is computed using retrospective analysis of dynamic programming method. The corresponding Bellman functional equation is:

$$f_t(\omega_t, V_t) = \lambda\omega_t - (1-\lambda)V_t \quad , \quad t = \overline{1,T}$$
$$f_{t-1}(\omega_{t-1}, V_{t-1}) = \min_{(x_t, z_t)} E\big(f_t(\omega_t, V_t)\big) \quad , \quad t = \overline{1,T}$$

To simplify the notation used to evaluate the individual risk of each asset at any moment of time, the following will be used:

$$q_{tj} = E\big(\left|R_{tj} - r_{tj}\right|\big), t = \overline{1,T}, j = \overline{1,n}$$

The problem which has to be solved for time stage T is

$$\begin{cases} min\big(\lambda\omega_T - (1-\lambda)E(V_T)\big) & \\ V_T = V_{T-1} + R_T x_T & \\ \omega_T = \omega_{T-1} + z_T & \\ E\big(\left|R_{Tj}x_{Tj} - r_{Tj}x_{Tj}\right|\big) \le z_T, & j = \overline{1,n} \\ V_{T-1} = \sum_{j=1}^{n} x_{Tj} & \\ x_{Tj} \ge 0, & j = \overline{1,n} \end{cases}$$

Assuming ω_{T-1} and V_{T-1} known and using Bellman functional equation, the following problem is obtained

$$\begin{cases}\min E\big(f_T(\omega_T, V_T)\big) = \min[\lambda(\omega_{T-1} + z_T) - (1-\lambda)(V_{T-1} + r_T X_T)] \\ q_T x_T \leq z_T \\ V_{T-1} = \sum_{j=1}^{n} x_{Tj} \\ x_{Tj} \geq 0, \qquad\qquad j = \overline{1,n}\end{cases}$$

this is similar to the problem of Cai.

To solve it, first all assets will be ranked according to their expected rate of return and the set of assets to be invested, denoted $\mathcal{F}$, is computed. Next, the amount to be invested in each asset is computed. By (x_T^*, z_T^*) is denoted the optimal solution of the problem. Replacing this solution into Bellman functional equation we obtain:

$$\begin{aligned} f_{T-1}(\omega_{T-1}, V_{T-1}) &= \min_{(x_T, z_T)} E\left(f_T(\omega_T, V_T)\right) \\ &= \lambda(\omega_{T-1} + z_T^*) - (1-\lambda)\left(V_{T-1} + \sum_{j=1}^{n} r_{Tj} x_{Tj}^*\right) \\ &= \lambda\left(\omega_{T-1} + V_{T-1}\left(\sum_{l\in\mathcal{F}} \frac{1}{q_{Tl}}\right)^{-1}\right) \\ &\quad - (1-\lambda)\left(V_{T-1} + \sum_{j=1}^{n} r_{Tj}\frac{V_{T-1}}{q_{Tj}}\left(\sum_{l\in\mathcal{F}}\frac{1}{q_{Tl}}\right)^{-1}\right) \\ &= \lambda(\omega_{T-1} + V_{T-1}a_T) - (1-\lambda)(V_{T-1} + V_{T-1}a_T b_T) \\ &= \lambda\omega_{T-1} + \lambda V_{T-1}a_T - (1-\lambda)V_{T-1} - (1-\lambda)V_{T-1}a_T b_T \\ &= \lambda\omega_{T-1} - V_{T-1}[(1-\lambda) + (1-\lambda)a_T b_T - \lambda a_T] \\ &= \lambda\omega_{T-1} - V_{T-1}[(1-\lambda)(1 + a_T b_T) - \lambda a_T] = \lambda\omega_{T-1} - V_{T-1}c_T \end{aligned}$$

where

$$\begin{aligned} a_T &= \left(\sum_{l\in\mathcal{F}} \frac{1}{q_{Tl}}\right)^{-1} \\ b_T &= \sum_{l\in\mathcal{F}} \frac{r_{Tl}}{q_{Tl}} \\ c_T &= [(1-\lambda)(1 + a_T b_T) - \lambda a_T] = c_{T+1}(1 + a_T b_T) - \lambda a_T \end{aligned}$$

Proceeding to the next stage *(T-1),* the problem to be solved is

$$\begin{cases} min\big(\lambda\omega_{T-1} - (1-\lambda)E(V_{T-1})\big) \\ V_{T-1} = V_{T-2} + R_{T-1}x_{T-1} \\ \omega_{T-1} = \omega_{T-2} + z_{T-1} \\ E\big(\big|R_{T-1,j}x_{T-1,j} - r_{T-1,j}x_{T-1,j}\big|\big) \le z_{T-1}, \quad j = \overline{1,n} \\ V_{T-2} = \sum_{j=1}^{n} x_{T-1,j} \\ x_{T-1,j} \ge 0, \qquad\qquad\qquad\qquad j = \overline{1,n} \end{cases}$$

Assuming ω_{T-2} and V_{T-2} known and using Bellman functional equation, together with some constraints from problem at time stage *T-1,* the objective function becomes

$$\begin{aligned} min\big(\lambda\omega_{T-1} - (1-\lambda)E(V_{T-1})\big) &= min\Big(E\big(f_{T-1}(\omega_{T-1}, V_{T-1})\big)\Big) \\ &= min(\lambda\omega_{T-1} - E(V_{T-1})c_T) \\ &= min\big(\lambda(\omega_{T-2} + z_{T-1}) + c_T(V_{T-2} + r_{T-1}x_{T-1})\big) \end{aligned}$$

and the parametric optimization problem to be solved is:

$$(PO(\lambda)\,2, T-1) \quad \begin{cases} \min[\lambda(\omega_{T-2} + z_{T-1}) + c_T(V_{T-2} + r_{T-1}x_{T-1})] \\ q_{T-1}x_{T-1} \le z_{T-1} \\ V_{T-2} = \sum_{j=1}^{n} x_{T-1,j} \\ x_{T-1,j} \ge 0, \qquad\qquad\qquad\qquad j = \overline{1,n} \end{cases}$$

The structure of this problem is somehow similar to that of Cai. The difference is given by c_T value, which in the model of Cai is replaced by $(1-\lambda)$.

The principle of solving this problem is similar to that of Cai, meaning that first, the set of assets to be invested is computed, and after that the amount to be invested in each asset is determined. The solution is presented in the following theorem.

Theorem 6 (solution for parametric optimization problem at time stage T-1) (see [40])

The optimal solution for parametric optimization problem $(PO(\lambda)\ 2\ , T-1)$ *is*

$$x^*_{T-1,j} = \begin{cases} \dfrac{V_{T-2}}{q_{T-1,j}} \left(\displaystyle\sum_{j \in \mathcal{F}} \frac{1}{q_{T-1,j}} \right)^{-1}, & j \in \mathcal{F} \\ 0, & j \notin \mathcal{F} \end{cases}$$

$$z^*_{T-1} = V_{T-2} \left(\sum_{j \in \mathcal{F}} \frac{1}{q_{T-1,j}} \right)^{-1}$$

where $\mathcal{F}$ *is the set of assets to be invested and is computed as follows:*

1. *If* $c_T > 0$ *(case similar to Cai), then:*
   if $\exists k = \overline{0, n-2}$ *such that:*

$$\frac{r_{T-1,n} - r_{T-1,n-1}}{q_{T-1,n}} < \frac{\lambda}{c_T}$$

$$\frac{r_{T-1,n} - r_{T-1,n-2}}{q_{T-1,n}} + \frac{r_{T-1,n-1} - r_{T-1,n-2}}{q_{T-1,n-1}} < \frac{\lambda}{c_T}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\frac{r_{T-1,n} - r_{T-1,n-k}}{q_{T-1,n}} + \frac{r_{T-1,n-1} - r_{T-1,n-k}}{q_{T-1,n-1}} + \cdots + \frac{r_{T-1,n-k+1} - r_{T-1,n-k}}{q_{T-1,n-k+1}} < \frac{\lambda}{c_T}$$

$$\frac{r_{T-1,n} - r_{T-1,n-k-1}}{q_{T-1,n}} + \frac{r_{T-1,n-1} - r_{T-1,n-k-1}}{q_{T-1,n-1}} + \cdots + \frac{r_{T-1,n-k} - r_{T-1,n-k-1}}{q_{T-1,n-k}} \geq \frac{\lambda}{c_T}$$

then $\mathcal{F} = \{n, n-1, \ldots, n-k\}$,
else $\mathcal{F} = \{n, n-1, \ldots, 1\}$.

2. *If* $c_T = 0$, *then* $\mathcal{F} = \{n, n-1, \ldots, 1\}$.

3. *If* $c_T < 0$, *then:*

   If $\exists k = \overline{0, n-2}$ *such that:*

$$\frac{r_{T-1,1}-r_{T-1,2}}{q_{T-1,1}} > \frac{\lambda}{c_T}$$
$$\frac{r_{T-1,1}-r_{T-1,3}}{q_{T-1,1}} + \frac{r_{T-1,2}-r_{T-1,3}}{q_{T-1,2}} > \frac{\lambda}{c_T}$$
$$\dots\dots\dots\dots$$
$$\frac{r_{T-1,1}-r_{T-1,n-k-1}}{q_{T-1,1}} + \frac{r_{T-1,2}-r_{T-1,n-k-1}}{q_{T-1,2}} + \cdots + \frac{r_{T-1,n-k-2}-r_{T-1,n-k-1}}{q_{T-1,n-k-2}} > \frac{\lambda}{c_T}$$
$$\frac{r_{T-1,1}-r_{T-1,n-k}}{q_{T-1,1}} + \frac{r_{T-1,2}-r_{T-1,n-k}}{q_{T-1,2}} + \cdots + \frac{r_{T-1,n-k-1}-r_{T-1,n-k}}{q_{T-1,n-k-1}} \le \frac{\lambda}{c_T}$$

then $\mathcal{F} = \{1,2,\dots,n-k-1\}$,
else $\mathcal{F} = \{1,2,\dots,n\}$.

Continuing the above algorithm for all time stages, the optimal solution is computed.

Total wealth at the end of time stage T is

$$\begin{aligned} E(V_T) &= V_{T-1} + r_T x_T^* = V_0 + r_1 x_1^* + r_2 x_2^* + \cdots + r_{T-1} x_{T-1}^* + r_T x_T^* \\ &= (V_0 + V_0 a_1 b_1) + r_2 x_2^* + \cdots + r_{T-1} x_{T-1}^* + r_T x_T^* \\ &= V_1 + V_1 a_2 b_2 + \cdots + r_{T-1} x_{T-1}^* + r_T x_T^* \\ &= V_0 (1 + a_1 b_1)(1 + a_2 b_2) \dots (1 + a_T b_T) \end{aligned}$$

and total risk for the investor is

$$\begin{aligned} \omega_T &= \omega_{T-1} + z_T = \omega_0 + z_1 + z_2 + \cdots + z_T = V_0 a_1 + z_2 + \cdots + z_T \\ &= V_0 a_1 + V_1 a_2 + \cdots + z_T \\ &= V_0 a_1 + V_0 (1 + a_1 b_1) a_2 + V_0 (1 + a_1 b_1)(1 + a_2 b_2) a_3 + \cdots \\ &+ V_0 (1 + a_1 b_1) \dots (1 + a_{T-1} b_{T-1}) a_T = \\ &= V_0 a_1 + \sum_{i=1}^{T-1} \left[a_{i+1} V_0 \prod_{j=1}^{i} \left(1 + a_j b_j\right) \right] \end{aligned}$$

The following diagram presents step by step the algorithm employed to compute the optimal solution:

1	t	T-1	T
$\begin{cases} min(\lambda\omega_1-(1-\lambda)E(V_1)) \\ V_1=V_0+R_1x_1 \\ \omega_1=\omega_0+z_1 \\ E(\lvert R_{1j}x_{1j}-r_{1j}x_{1j}\rvert)\le z_1, \quad j=\overline{1,n} \\ V_0=\sum_{j=1}^{n}x_{1j} \\ x_{1j}\ge 0, \quad j=\overline{1,n} \end{cases}$	$\begin{cases} min(\lambda\omega_t-(1-\lambda)E(V_t)) \\ V_t=V_{t-1}+R_tx_t \\ \omega_t=\omega_{t-1}+z_t \\ E(\lvert R_{tj}x_{tj}-r_{tj}x_{tj}\rvert)\le z_t, \quad j=\overline{1,n} \\ V_{t-1}=\sum_{j=1}^{n}x_{tj} \\ x_{tj}\ge 0 \quad j=\overline{1,n} \end{cases}$	$\begin{cases} min(\lambda\omega_{T-1}-(1-\lambda)E(V_{T-1})) \\ V_{T-1}=V_{T-2}+R_{T-1}x_{T-1} \\ \omega_{T-1}=\omega_{T-2}+z_{T-1} \\ E(\lvert R_{T-1,j}x_{T-1,j}-r_{T-1,j}x_{T-1,j}\rvert)\le z_{T-1} \quad j=\overline{1,n} \\ V_{T-2}=\sum_{j=1}^{n}x_{T-1,j} \\ x_{T-1,j}\ge 0 \quad j=\overline{1,n} \end{cases}$	$\begin{cases} min(\lambda\omega_T-(1-\lambda)E(V_T)) \\ V_T=V_{T-1}+R_Tx_T \\ \omega_T=\omega_{T-1}+z_T \\ E(\lvert R_{Tj}x_{Tj}-r_{Tj}x_{Tj}\rvert)\le z_T \quad j=\overline{1,n} \\ V_{T-1}=\sum_{j=1}^{n}x_{Tj} \\ x_{Tj}\ge 0 \quad j=\overline{1,n} \end{cases}$
$\begin{cases} min\left(E(f_1(\omega_1,V_1))\right)= \\ =min(\lambda(\omega_0+z_1)-c_2(V_0+r_1x_1)) \\ q_1x_1\le z_1 \\ V_0=\sum_{j=1}^{n}x_{1j} \\ x_{1j}\ge 0, \quad j=\overline{1,n} \end{cases}$	$\begin{cases} min\left(E(f_t(\omega_t,V_t))\right)= \\ =min(\lambda(\omega_{t-1}+z_t)-c_{t+1}(V_{t-1}+r_tx_t)) \\ q_tx_t\le z_t \\ V_{t-1}=\sum_{j=1}^{n}x_{tj} \\ x_{tj}\ge 0, \quad j=\overline{1,n} \end{cases}$	$\begin{cases} min\left(E(f_{T-1}(\omega_{T-1},V_{T-1}))\right)= \\ =min(\lambda(\omega_{T-2}+z_{T-1})-c_T(V_{T-2}+r_{T-1}x_{T-1})) \\ q_{T-1}x_{T-1}\le z_{T-1} \\ V_{T-2}=\sum_{j=1}^{n}x_{T-1,j} \\ x_{T-1,j}\ge 0, \quad j=\overline{1,n} \end{cases}$	$\begin{cases} min\left(E(f_T(\omega_T,V_T))\right)= \\ =min(\lambda(\omega_{T-1}+z_T)-(1-\lambda)(V_{T-1}+r_Tx_T)) \\ q_Tx_T\le z_T \\ V_{T-1}=\sum_{j=1}^{n}x_{Tj} \\ x_{Tj}\ge 0, \quad j=\overline{1,n} \end{cases}$
$\mathcal{F}=\begin{cases} \ldots & c_2<0 \\ \ldots & c_2=0 \\ \ldots & c_2>0 \end{cases}$; $x_1^*;z_1^*$	$\mathcal{F}=\begin{cases} \ldots & c_{t+1}<0 \\ \ldots & c_{t+1}=0 \\ \ldots & c_{t+1}>0 \end{cases}$; $x_t^*;z_t^*$	$\mathcal{F}=\begin{cases} \ldots & c_T<0 \\ \ldots & c_T=0 \\ \ldots & c_T>0 \end{cases}$; $x_{T-1}^*;z_{T-1}^*$	$\mathcal{F}=<$; $x_T^*;z_T^*$
	$f_{t-1}(\omega_{t-1},V_{t-1})=\min_{(x_t,z_t)}E\left(f_t(\omega_t,V_t)\right)=\lambda\omega_{t-1}-V_{t-1}c_t$	$f_{T-2}(\omega_{T-2},V_{T-2})=\min_{(x_{T-1},z_{T-1})}E\left(f_{T-1}(\omega_{T-1},V_{T-1})\right)=\lambda\omega_{T-2}-V_{T-2}c_{T-1}$	$f_{T-1}(\omega_{T-1},V_{T-1})=\min_{(x_T,z_T)}E\left(f_T(\omega_T,V_T)\right)=\lambda\omega_{T-1}-V_{T-1}c_T$
	$a_t;b_t;c_t=fct(a_t;b_t;c_{t+1})$	$a_{T-1};b_{T-1};c_{T-1}=fct(a_{T-1};b_{T-1};c_T)$	$a_T;b_T;c_T=fct(a_T;b_T;(1-\lambda))$

Young model

The minimax model created by Robert Young (see [37]) maximizes the minimum over all time stages of expected return, subject to restrictions that average return of portfolio exceeds a predefined level and the total amount invested at each time stage does not exceed the available amount. The mathematical model of the problem is presented below

$$\max_{j=\overline{1,n}}\left\{\min_{t=\overline{1,T}}\sum_{j=1}^{n}x_{tj}r_{tj}\right\}$$

$$\sum_{j=1}^{n}x_j\overline{r_j} \geq G$$

$$\sum_{j=1}^{n}x_{tj} \leq V_{t-1}$$

$$x_{tj} \geq 0 \qquad t=\overline{1,T}, j=\overline{1,n}$$

where

$$\overline{r_j}=\frac{1}{T}\sum_{t=1}^{T}r_{tj}$$

G is the predefined level for average return of portfolio.

6. Conclusions

Analyzing the Mean-Variance Model and its extensions we can conclude that from practical point of view the minimax model is the easiest to be implemented, because the analytical solution is computed with low effort. This model, like all others from Portfolio Theory, has a high sensitivity for mean. We consider that this model fits to our goal (energy optimization) and we intend to implement it in our future research project.

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