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# DISPERSION EQUATION FOR PLASMA WAVE PROPAGATION AT THE INTERFACE OF A STOCHASTIC ENVINRONMENT

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**ABSTRACT**. The present paper investigates the effect of a stochastic term in the equation of motion in the MHD approximation. The end purpose is to find whether or not, for the case of an interface between two media of different physical properties, the stability behavior of the waves characteristic to the interface changes. Our results show that, for a certain parameter set, the stability behavior does change: namely a configuration which is stable in the deterministic approach may become unstable when stochastic effects are considered.

***Keywords:*** *MHD waves, Stochastic processes, instabilities*

## 1. SHORT INTRODUCTION IN THE PHYSICS OF STOCHASTIC PROCESSES

The archetypical **Brownian Motion** [4] consists of a massive particle subjected to two types of forces: a frictional force, that dissipates the kinetic energy of the particle and a random force that pushes the particle in an erratic way. This random force stands for the effects of the interactions (collisions) of the particle with all the other particles in the system. The characteristics

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(mean value and variance) of the random term are determined by the energy content available in the system. The archetypical Brownian motion assumes that the many particle system is in contact with a heat reservoir. However, the energy content of the system might also be given by the presence of a magnetic field and so on. The development of the Brownian motion toy model was necessary in order to be able to perform analytical treatment on systems of many coupled interacting elements. Insight to this problem may be gained by using inferences about the statistical behaviour of the particle interaction. One studies the motion of just one particle and considers that the influence of all the other particles is given by the action of a stochastic force with certain statistical properties. As such, we will only have access to information regarding macroscopic characteristics of the underlying processes.

The system under analysis consists of *N* Brownian particles, "living" in an isotropic 3 dimensional coordinate space, of a finite volume and moving with friction. Each Brownian particle is characterized by a mass $m$, position $r$ and velocity v. Due to friction, one part of the deterministic force will be given by $-\nu$v, with $\nu$ the friction coefficient acting on the particle. The rest of the deterministic force can be derived from a potential $U\ (r1, .., rN\ )$. The fluctuations in velocity space are given by realizations of a Gaussian-like process (i.e. they are Gaussian random variables), characterized by diffusion coefficients $B$. At thermodynamic equilibrium the values of the diffusion coefficients are constrained by the value of the temperature of the system. The velocity is then itself a stochastic variable. For a zero external potential, historically, this velocity is called Brownian motion. The general mathematical name is Wiener process.

Taking the above considerations into account, the A-Langevin equation describing the system may be written as

$$\frac{d\mathbf{v}}{dt} = -\nu\mathbf{v} - \frac{1}{m}\nabla U(\mathbf{r}_1, .., \mathbf{r}_N) + \sqrt{2B}\vec{A}_t(t), \tag{1.1}$$

where

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \tag{1.2}$$

and $\vec{A}_t$ is white noise of parameter $t$ with statistical properties

$$\langle \vec{A}_t(t) \rangle = 0, \tag{1.3}$$

$$\langle A_{ti}(t) A_{tj}(t') \rangle = \sigma \delta_{ij} \delta(t - t'). \tag{1.4}$$

A property of white noise is that its derivative is the Wiener process $\beta_t$

$$A_t = \frac{d\beta_t}{dt} \tag{1.5}$$

where the properties of the Wiener process are

- $\beta_0 = 0$
- $\beta_t$ is almost surely continuous
- $\beta_t$ has independent increments
- $\beta_t - \beta_s \in \mathcal{N}(0, t - s)$.

## 2. POSING OF THE PROBLEM AND ALGORITHM

The common feature of all types of classical wave analysis is the fact that the adopted mathematical model of the phenomenon is deterministic. It turns out, that – just as in investigation of many other phenomena – deterministic modelling of waves does not always result in an adequate presentation of reality. Because of the existence of many uncontrolled factors determining real wave processes a stochastic description, i.e., one based on probability theory, is often more adequate. A stochastic nature of real wave processes results above all from the inhomogeneity and uncertainty of the structure of most wave - transmitting media. In order to account for the complexity of real media one usually introduces a mathematical model called a stochastic or random medium. In general, this means a medium whose properties are described in terms of probability theory, or, more specifically, by means of random functions $A(\vec{r}, t)$ of position and time. *Temporal variations of the properties of the medium are often very slow; consequently, we mostly use a spatial random field* $A(\vec{r}, t)$ which can take scalar, vector or tensor values depending on the particular situation. A random field $A(\vec{r}, t)$ may possess continuously and smoothly varying sample functions, or it may just be a discrete - valued random function [3].

The purpose of this work is to study the effect of a stochastic force on wave propagation in ionized plasmas. The route undertaken is to work in the MHD framework in an A-Langevin equation approach, i.e., to include a stochastic term in the equation of motion.

The general approximation is that the stochastic term is seen as constant by the fluid, i.e., the characteristic timescale of the noise is much smaller than any characteristic timescale of the fluids, such that $\vec{A} = \vec{A}_z(z)$.

## 3. GOVERNING EQUATIONS AND BASIC ASSUMPTIONS

We consider the case of the interface between two media; particular numerical values are taken for an interface between the solar corona and a prominence. We will follow closely the derivation in [1].

The starting linearized equations are

$$\nabla \cdot \vec{v}_i = 0, \quad \nabla \cdot \vec{b}_i = 0 \quad i \in \{1, 2\}, \tag{3.1}$$

for the perturbed velocity and magnetic field in both regions. The momentum equations are

$$\rho_1 \frac{\partial \vec{v}_1}{\partial t} = -\nabla P_1 + \frac{B_{01}}{\mu_0} \frac{\partial \vec{b}_1}{\partial x} + \rho_1 \mathcal{V} + \vec{A}_1, \tag{3.2}$$

$$\rho_2 \frac{\partial \vec{v}_2}{\partial t} + \rho_2 v_0 \frac{\partial \vec{v}_2}{\partial x} = -\nabla P_2 + \frac{B_{02}}{\mu_0} \frac{\partial \vec{b}_2}{\partial x} + \vec{A}_2, \tag{3.3}$$

where $\vec{A}_i = \left(A_{ix}, A_{iy}, A_{iz}\right) = \imath \vec{W}_z e^{\imath(k_x x - \omega t)}$ and $\vec{W}_z$ is Gaussian white noise in argument $z$

$$\langle \vec{W}_z(z) \rangle = 0, \quad \langle W_{zi}(z) W_{zj}(z') \rangle = \sigma \delta_{zz'} \delta_{ij}. \tag{3.4}$$

The term $\vec{A}$ has complex amplitude in our approach because we will use these equations to perform the usual normal mode analysis. Since the noise term is akin to the friction term and we know that friction will lead to an imaginary component in the dispersion equation, we make sure that the noise term will have a similar kind of contribution by making the amplitude of $A$ complex.

The induction equations are

$$\frac{\partial \vec{b}_1}{\partial t} = B_{01}\frac{\partial \vec{v}_1}{\partial x}, \tag{3.5}$$

$$\frac{\partial \vec{b}_2}{\partial t} + v_0\frac{\partial \vec{b}_2}{\partial x} = B_{02}\frac{\partial \vec{v}_2}{\partial x} + \mathcal{R}, \tag{3.6}$$

where

$$\mathcal{V} = 3\nu\left[\tilde{\vec{b}}\left(\tilde{\vec{b}}\cdot\nabla\right) - \frac{1}{3}\nabla\right]\left[\tilde{\vec{b}}\cdot(\tilde{\vec{b}}\cdot\vec{v}_1)\right], \quad \tilde{\vec{b}} = \vec{b}/|\vec{b}|, \tag{3.7}$$

$$\mathcal{R} = \eta\nabla^2\vec{b}_2 + \frac{(\eta_c - \eta)}{|\vec{B}_0^2|}\nabla\times\left\{\left[(\nabla\times\vec{b}_2)\times\vec{B}_0\right]\times\vec{B}_0\right\}. \tag{3.8}$$

**Jump condition at the interface**

Imposing the jump condition is done as follows: consider the z component of the equation of motion, denoting it by $\alpha$ . Use the approximation of one dimensional steady state $\nabla\rightarrow\hat{z}\frac{d}{dz}$ and $\partial/\partial t = 0$ and the two equations of motion become

$$\frac{d}{dz}\alpha = 0 \tag{3.9}$$

where

$$\begin{aligned}\alpha &= -P_1 + \rho_1\nu\frac{\partial v_z}{\partial z} + \imath\beta_{z1}e^{\imath(k_x x-\omega t)} \text{ for region 1} \qquad (3.10)\\ &= -P_2 + \beta_{z2}e^{\imath(k_x x-\omega t)} \text{ for region 2}\end{aligned}$$

Integrating across the surface, from region 1 to region 2

$$\int_1^2 \frac{d\alpha}{dz} dz = 0 \text{ yields } \alpha(2) = \alpha(1) \tag{3.11}$$

i.e.

$$-P_1 + \rho_1 \nu \frac{\partial v_z}{\partial z} + \imath \beta_{z1} e^{\imath(k_x x - \omega t)} = -P_2 + \imath \beta_{z2} e^{\imath(k_x x - \omega t)}. \tag{3.12}$$

**Normal mode analysis**

The usual procedure at this point in MHD wave theory is to perform normal mode analysis (NMA). This means looking for solutions of some imposed type, where the time behavior is swept under the rug as an exponential $\sim e^{-\imath wt}$. Normal mode analysis does not use the extra step of integrating that the Fourier analysis uses.

For illustration purposes, consider the Langevin equation

$$\frac{dv}{dt} = -\nu v + \tilde{A}, \tag{3.13}$$

where $\tilde{A}$ is a noise term of known statistical properties. Decide that you want to perform NMA for this equation. That means that you will look for a solution of the type $v = \hat{v}(z)\ exp\{\imath(k_x x - \omega t)\}$. The equation then becomes

$$-\imath\omega\hat{v} \exp\{\imath(k_x x - \omega t)\} = -\nu\hat{v} \exp\{\imath(k_x x - \omega t)\} + A, \tag{3.14}$$

and thus

$$-\imath\omega\hat{v} = -\nu\hat{v} + A \exp\{-\imath(k_x x - \omega t)\}. \tag{3.15}$$

The term $A\ exp\{-\imath(k_x x - \omega t)\}$ is a noise term, of known statistical properties. We may denote it by some other symbol and carry on the usual calculations. Note that when we will apply this approach to the MHD equations ofmotion, $\vec{A}\ exp\{-\imath(k_x x - \omega t)\}$= $\imath\overrightarrow{W}_z$.

For the more complex set of MHD equations, we proceed exactly as in [1]. Technically, we will obtain the dispersion equation for the case in which the equations of motion are not homogeneous.

Because the mean of the white noise is zero, we obtain, just as in [1] the expressions for the two pressures

$$P_1 = \zeta\frac{\rho_1}{k_x}\left(D_{A1} - \frac{\imath}{2}k_x^2\nu\omega\right) \tag{3.16}$$

$$P_2 = -\zeta\frac{\rho_2 D_{A2}}{k_x}\left(1 + \frac{\imath}{2}\frac{k_x^2\eta_C}{\Omega}\right). \tag{3.17}$$

The jump condition in the NMA assumption (using the expressions for the pressure Eqs. (3.16)-(3.17)) becomes

$$-\imath\beta_{z1}+\zeta\left(\frac{D_{A1}}{k_x}\rho_1 + \frac{\imath}{2}k_x\nu\rho_1\omega\right) = -\imath\beta_{z2}-\zeta\left(\frac{D_{A2}}{k_x}\rho_2 + \imath\frac{D_{A2}k_x\eta_C\rho_2}{2\Omega}\right). \tag{3.18}$$

The result obtained for the dispersion equation is

$$D(\omega) = D_r + \imath D_i = 0, \tag{3.19}$$

where in this case

$$D_r = D_{A1} + dD_{A2}, \tag{3.20}$$

and it does not contain a stochastic term, and

$$D_i = \beta + \frac{1}{2}\left(k_x^2\nu\omega + \frac{dD_{A2}k_x^2\eta_c}{\Omega}\right), \quad \text{with } \beta = \frac{\beta_{z2}-\beta_{z1}}{\zeta_0}, \tag{3.21}$$

where $\zeta_0$ is the perturbation at the interface calculated for some typical scale of the plasma, and the stochastic term is beta.

## 4. RESULTS

We now follow the algorithm in [2] and find $\omega_0$ as a solution of $D_r = 0$. This is a second order algebraical equation which produces two solutions, $\omega_0^+$ and $\omega_0^-$. These are identical to the ones obtained in the deterministic case, since there is no stochastic term in $D_r$.

Further, it is assumed that the terms in $D_i$ produce a perturbation $\delta\omega$ and as such $\delta\omega$ is obtained by a Taylor expansion of $D(\omega)$ for $\omega = \omega_0 + \delta\omega$; it will thus present with two solutions (for each of the $\omega_0$): $\delta\omega_+$ and $\delta\omega_-$.

As shown in [2],

$$\delta\omega(\omega_0) = -\left(\frac{1}{\partial D_r(\omega)/\partial\omega} D_i(\omega)\right)\Big|_{\omega=\omega_0}, \tag{4.1}$$

and so

$$\delta\omega(\omega_0) = -\frac{\left[2\beta + k_x^2\nu\omega + \frac{dk_x^2\eta_c(-k_x^2 v_{A2}^2 + (-k_x v_0 + \omega)^2)}{-k_x v_0 + \omega}\right]}{4\omega + 4d(-k_x v_0 + \omega)}\Bigg|_{\omega=\omega_0}. \tag{4.2}$$

We checked that at all times the ratio between the perturbation frequency and $\omega_0$ is very small, of the order $10^{-5}$ (Figure 1).

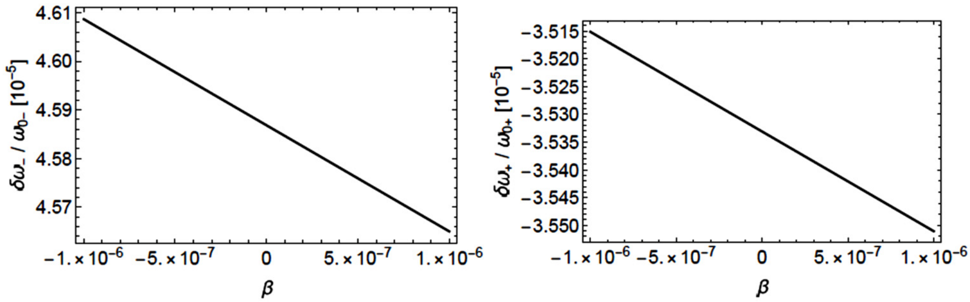


**Figure 1.** Ratio of the frequency $\delta\omega$ to its corresponding $\omega_0$.

If the $\delta\omega$ are positive, an instability appears.

In the limit $\beta \to 0$ the classical result is recovered, in which $\delta\omega$ is always negative. However, allowing for a nonzero $\beta$ leads to positive values of the $\delta\omega$, and thus the appearance of instability.

Analytically, the condition that $\delta\omega > 0$, i.e., there is instability, has the form

$$\beta > \frac{1}{8}\left[\omega + d\Omega - 4k_x^2\omega\nu + \frac{4d_x^2\eta_C D_{A2}}{\Omega}\right]\Bigg|_{\omega=\omega_0}. \tag{4.3}$$

So for each of the two cases

$$\delta\omega_- > 0 \text{ if } \beta > \frac{1}{8}\left[\omega + d\Omega - 4k_x^2\omega\nu + \frac{4d_x^2\eta_C D_{A2}}{\Omega}\right]\Bigg|_{\omega=\omega_{0-}}. \quad (4.4)$$

$$\delta\omega_+ > 0 \text{ if } \beta > \frac{1}{8}\left[\omega + d\Omega - 4k_x^2\omega\nu + \frac{4d_x^2\eta_C D_{A2}}{\Omega}\right]\Bigg|_{\omega=\omega_{0+}}. \quad (4.5)$$

For $\beta \to 0$, the deterministic case of [1] is recovered and the $\delta\omega$ frequencies are negative. Since in the absence of noise, the frequencies are always negative, the mathematical result is that the presence of a stochastic component is a source of instability. The condition for instability says that there will be instability at the interface provided that the noise difference on the two sides of the interface exceeds a certain threshold.

This can be seen for both solutions of the $\delta\omega$ in Figure 2, for parameter values used in [1].

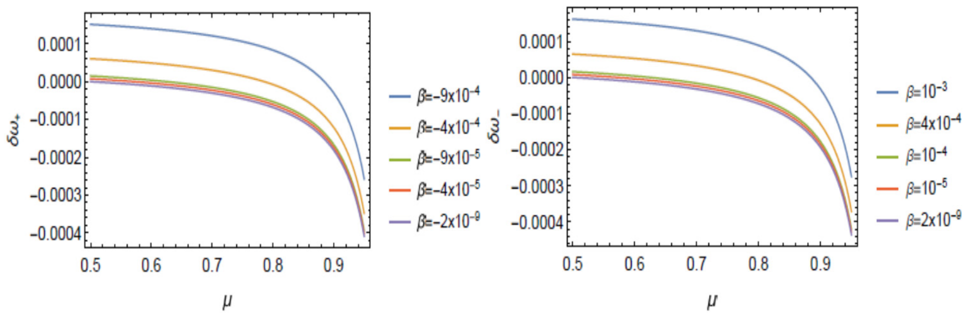


**Figure 2.** Variation of $\delta\omega$ as a function of the ionization parameter $\mu$, for different values of the parameter $\beta$, which quantifies the influence of the noise. The numerical values used to obtain the plots are: $T_2 = 10^4$, $\nu = 10^{-10}$, $k_x = 5 \cdot 10^{-6}$, $\rho_2 = 5 \cdot 10^{-11}$, $v_{a1} = 315000$, $v_{a2} = 28000$, $v_0 = 20000$, $d = \sqrt{10}$.

**Forward mode**

Let us focus on the forward mode, i.e., the one identified by $\omega_{0+}$ and $\delta\omega_+$. The results obtained for the deterministic case are shown in Figure 3. It is always negative, i.e., stable. But as seen in Figure 2 left, there are combinations of $\{\mu, \beta\}$ for which $\delta\omega_+$ becomes positive.

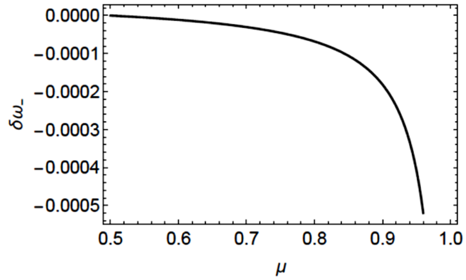


**Figure 3.** Variation of $\delta\omega$ in the forward wave case, as a function of the ionization parameter $\mu$, for $\beta = 0$.

## 5. CONCLUSIONS

In the present paper the MHD approximation was considered for the case of an interface between two plasmas with different properties. A stochastic term was allowed for in the equation of motion. With the purpose of obtaining a dispersion equation for the waves present at the interface, the jump condition between the two media was obtained. Numerical implementation of the dispersion equation indeed shows that a stochastic term might change the stability behavior of the system.

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